Confluence for topological rewriting systems

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I. INTRODUCTION

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

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- ➔ Term rewriting
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In computer algebra

- Polynomial reduction
- Involutive divisions

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Abstraction

Abstract rewriting theory

Abstract properties common to all concrete rewriting systems: **termination**, **confluence**, **normal forms**

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- $\rightarrow$ a binary relation on A

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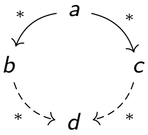
Confluence



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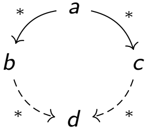


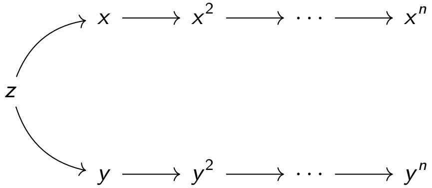
Example

Multivariate division with respect to R is **confluent** iff R is a **Gröbner basis**

Confluence "at the limit"

In $\mathbb{K}[[x, y, z]]$ with the inverse deglex order such that $z > y > x$ take

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$



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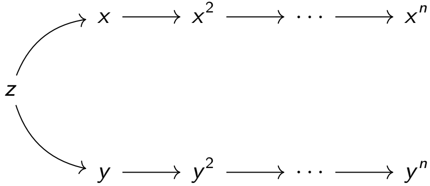

The two branches will never have a common element
Hence the system is **not confluent**

However with the (x, y, z)-adic topology both branches converge to 0

Topological Abstract Rewriting System

- ➔ (X, τ) a **topological space**
- ➔ $\to$ a binary relation on X

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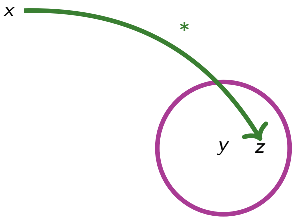
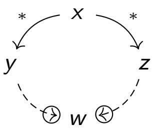
Topological rewriting relation

Write $x \rightarrow\!\!\!\!\!\bigcirc\, y$ if for **every neighbourhood** U of y **there exists** $z \in U$ s.t. $x \xrightarrow{*} z$



Note how $x \xrightarrow{*} y$ implies $x \rightarrow\!\!\!\!\!\bigcirc\, y$

Topological confluence



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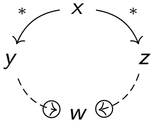
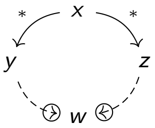


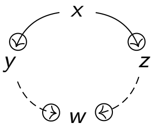
Theorem. [Chenavier 2020]

Standard basis ⇔ **topological confluence**
where **standard bases** are to formal power series as **Gröbner bases** are to polynomials

Topological confluence



Infinitary confluence



Theorem. [Chenavier 2020]
Standard basis $\Leftrightarrow$ **topological confluence** where **standard bases** are to formal power series as **Gröbner bases** are to polynomials

Of interest in computer science: **infinitary λ/Σ-terms**

Strength of confluences

For every TARS we have:

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For instance, if τ is the discrete topology, then $(X, \tau, \rightarrow)$ has **discrete rewriting**.

Counter-example of topological confluence ⇒ confluence

Consider again, in $\mathbb{K}[[x, y, z]]$

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- ➔ if $f \in I(R)$ then f has no constant coefficient

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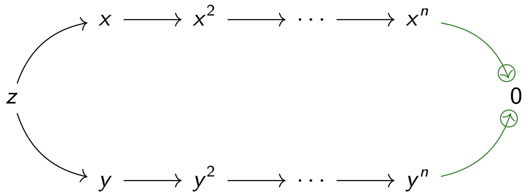
Thus the system is **topologically confluent**



However we saw previously that it is **not confluent**

Line with two origins

$$X := (\mathbb{R} \times \{\pm 1\}) \,/ \sim$$

where $(x, 1) \sim (x, -1)$ if $x \neq 0$

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Cyclic relation

$$X := [0, 2] \subset \mathbb{R}$$

$$\frac{1}{2^{n+1}} \rightleftarrows \frac{1}{2^n} \qquad 2 - \frac{1}{2^n} \rightleftarrows 2 - \frac{1}{2^{n+1}}$$

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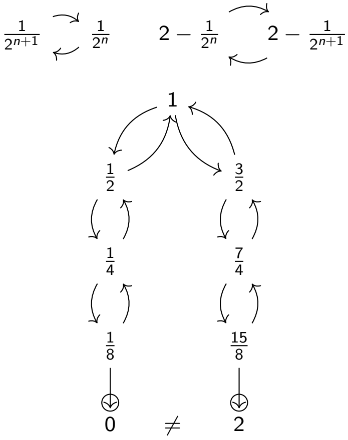

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$$\begin{array}{ccccc} & & (0,0) & & \\ & \swarrow & & \searrow & \\ (1,0) & & & & (0,1) \\ \downarrow & & & & \downarrow \\ (2,0) & & & & (0,2) \\ \Downarrow & & & & \Downarrow \\ (\infty, 0) & & \neq & & (0, \infty) \end{array}$$

Theorem. [Chenavier, Cluzeau, ML, 2024]

Let R be a set of formal power series and $<$ be a local monomial order that is compatible with the degree.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

II. EQUIVALENCE OF CONFLUENCES

Valuation

$$\mathrm{val}\left(xy^2z^2 + z^3 + y\right) = 1$$
$$\mathrm{val}\left(x^2yz + xy^2z\right) = 4$$

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Example of a convergent sequence

In $\mathbb{K}[[x, y, z]]$ the sequence (f_n) of powers of a variable (say x) converges: $\lim_{n\to\infty} f_n = 0$ because $\operatorname{val}(x^n - 0) \underset{n\to\infty}{\longrightarrow} \infty$

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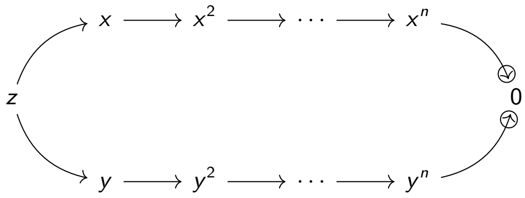
Hence in the example of the introduction:



Monomial orders

➔ Total order compatible with monomial multiplication

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Consequence: if $<$ is a **local** order **compatible with the degree** then

$$\mathrm{val}\,(f) = \deg\,(\mathrm{LM}\,(f))$$

Ideals of formal power series are topologically closed

➔ $\mathbb{K}[[x_1, \cdots, x_n]]$: local noetherian topological ring with respect to the $(x_1, \cdots, x_n)$-adic topology. Therefore a **Zariski ring** [Samuel, Zariski, 1975]

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- ➔ Constructive proof providing a **cofactor representation** of a formal power series in the topological closure of the ideal **[Chenavier, Cluzeau, ML, 2024]**

Proposition. For all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f \to_{\oplus} g$ then $f - g \in I$

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But I is topologically closed, hence $f - g \in I$

Theorem. [Chenavier, Cluzeau, ML, 2024]

Let R be a set of formal power series and $<$ be a **local** monomial order that is **compatible with the degree**.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

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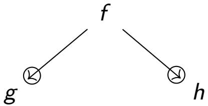

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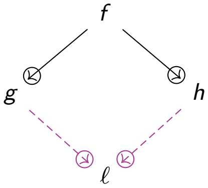

Close the diagram

- Fix R a non-empty set of non-zero formal power series
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- Write $\to$ the one-step rewriting relation induced by R and $<$

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Assume that $\rightarrow$ is **topologically confluent** *i.e.* R is a **standard basis** with respect to $<$ of the ideal $I := I(R)$ generated by R

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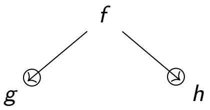
Let $f, g, h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that:

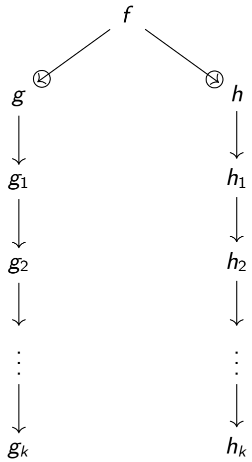


Goal

Construct inductively **two rewriting sequences** starting from g and h respectively that will be proven to be **Cauchy**

It will turn out that the limits are then equal and hence give a **common topological successor** to g and h

➔ By induction:
$\exists g \xrightarrow{*} g_k$ and $\exists h \xrightarrow{*} h_k$



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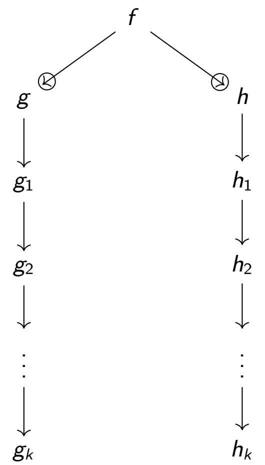
- If $g_k = h_k$, then it's over!



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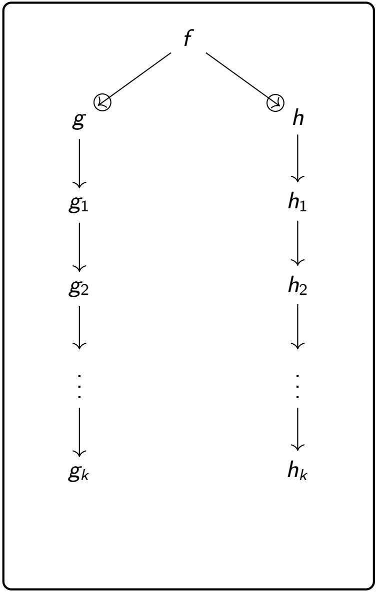
- From the previous proposition:

$$g_k - h_k \in I$$



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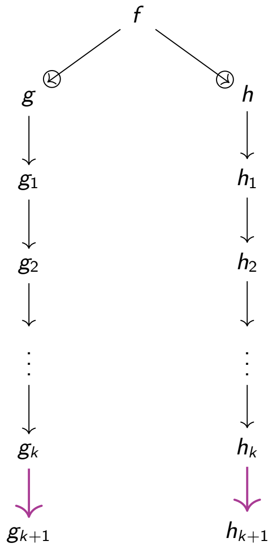
- ➔ From the previous proposition:

$$g_k - h_k \in I$$

- ➔ Rewrite LM $(g_k - h_k)$



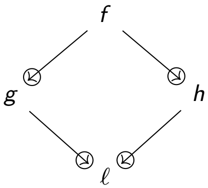
Facts

- ➔ the sequences $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ are **Cauchy**
- ➔ their limits are **equal**

Facts

- the sequences $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ are **Cauchy**
- their limits are **equal**

So $\lim_{k\to\infty} g_k = \lim_{k\to\infty} h_k =: \ell$



Which shows that $\to$ is **infinitary confluent**

III. CONCLUSION AND PERSPECTIVES

Conclusion and perspectives

Summary of presented notions and results:

- ▷ we introduced different confluence properties for topological rewriting systems
- ▷ we provided counter-examples for converse strength implications
- ▷ we showed that **topological confluence** is equivalent to **infinitary confluence** for formal power series thanks to the topological closure of ideals

Further works:

- ▷ study abstract properties of topological rewriting systems (*e.g.* C-R property, Newman's Lemma, etc ...)
- ▷ show that the topological rewriting relation induces convergent rewriting chains in the context of formal power series
- ▷ develop computational tools to study Taylor series and formal solutions of PDEs

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THANK YOU FOR LISTENING!