Topological closure of ideals of commutative formal power series and applications

Adya Musson-Leymarie

with **Cyrille Chenavier** and **Thomas Cluzeau**

Séminaire Calcul Formel - XLIM

June 6th, 2024

I. Introduction

- ▷ Rewriting on topological structures
- ▷ Formal power series as Cauchy-completion of polynomials

II. Ideals of formal power series

- ▷ Standard bases and topological confluence
- ▷ Topological closure of ideals

III. Applications

- ▷ Equivalence of confluences for formal power series
- ▷ Relation between standard bases and topological confluence

IV. Conclusion and perspectives

I. INTRODUCTION

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

Instances

In computer science

- ➔ Term rewriting
- ➔ β-reduction in λ-calculus

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

Instances

In computer science

- ➔ Term rewriting
- ➔ β-reduction in λ-calculus

In computer algebra

- ➔ Polynomial reduction
- ➔ Involutive divisions

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

Instances

In computer science

- ➔ Term rewriting
- ➔ β-reduction in λ-calculus

In computer algebra

- ➔ Polynomial reduction
- ➔ Involutive divisions

Abstraction

Abstract rewriting theory

Abstract properties common to all concrete rewriting systems: **termination**, **confluence**, **normal forms**

Abstract Rewriting System

- A an underlying set
- $\rightarrow$ a binary relation on A

We write $a \rightarrow b$ for $(a, b) \in \rightarrow$

Abstract Rewriting System

- A an underlying set
- $\rightarrow$ a binary relation on A

We write $a \rightarrow b$ for $(a, b) \in \rightarrow$

Transitive reflexive closure

We write $a \overset{*}{\rightarrow} b$ to express that $a = a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_\ell = b$

Abstract Rewriting System

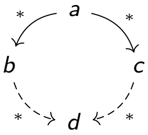
- A an underlying set
- $\to$ a binary relation on A

We write $a \to b$ for $(a, b) \in \to$

Transitive reflexive closure

We write $a \stackrel{*}{\to} b$ to express that $a = a_0 \to a_1 \to \cdots \to a_\ell = b$

Confluence



Abstract Rewriting System

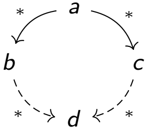
- A an underlying set
- $\to$ a binary relation on A

We write $a \to b$ for $(a, b) \in \to$

Transitive reflexive closure

We write $a \xrightarrow{*} b$ to express that $a = a_0 \to a_1 \to \cdots \to a_\ell = b$

Confluence



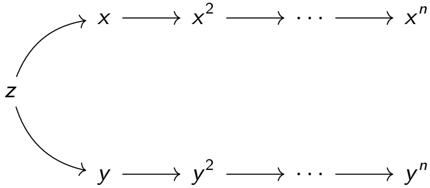
Example

Multivariate division with respect to R is **confluent** iff R is a **Gröbner basis**

Confluence “at the limit”

In $\mathbb{K}[[x, y, z]]$ with the inverse deglex order such that $z > y > x$ take

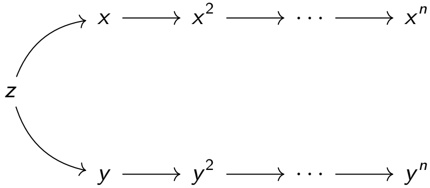
$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$



Confluence "at the limit"

In $\mathbb{K}[[x, y, z]]$ with the inverse deglex order such that $z > y > x$ take

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$



The two branches will never have a common element
Hence the system is **not confluent**

However with the (x, y, z)-adic topology both branches converge to 0

Topological Abstract Rewriting System

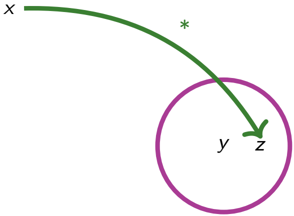
- ➔ (X, τ) a **topological space**
- ➔ $\rightarrow$ a binary relation on X

Topological Abstract Rewriting System

- (X, τ) a **topological space**
- $\to$ a binary relation on X

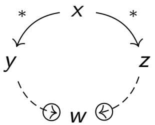
Topological rewriting relation

Write $x \mathrel{-\!\!\!\bigcirc\!\!\!\!\!\to} y$ if for **every neighbourhood** U of y **there exists** $z \in U$ s.t. $x \xrightarrow{*} z$

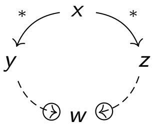


Note how $x \xrightarrow{*} y$ implies $x \mathrel{-\!\!\!\bigcirc\!\!\!\!\!\to} y$

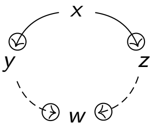
Topological confluence



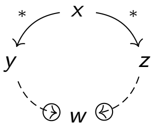
Topological confluence



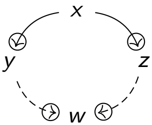
Infinitary confluence



Topological confluence



Infinitary confluence



Strength of confluences

For every TARS we have:
confluence $\Longrightarrow$ topological confluence
infinitary confluence $\Longrightarrow$ topological confluence

Discrete rewriting system

If $x \mathrel{-\!\!\!\circledcirc}\, y$ implies $x \xrightarrow{*} y$, then we say that the TARS $(X, \tau, \to)$ has **discrete rewriting**.

Discrete rewriting system

If $x \twoheadrightarrow y$ implies $x \xrightarrow{*} y$, then we say that the TARS $(X, \tau, \to)$ has **discrete rewriting**.

In such a case, **confluence**, **topological confluence** and **infinitary confluence** are **trivially equivalent**.

Discrete rewriting system

If $x \twoheadrightarrow y$ implies $x \xrightarrow{*} y$, then we say that the TARS $(X, \tau, \to)$ has **discrete rewriting**.

In such a case, **confluence**, **topological confluence** and **infinitary confluence** are **trivially equivalent**.

For instance, if τ is the **discrete topology**, then $(X, \tau, \to)$ has **discrete rewriting**.

Line with two origins

$$X := (\mathbb{R} \times \{\pm 1\}) \,/ \sim$$

where $(x, 1) \sim (x, -1)$ if $x \neq 0$

$$\forall n \in \mathbb{N}, \quad \left(\frac{1}{2^n}, 1\right) \to \left(\frac{1}{2^{n+1}}, 1\right)$$

Line with two origins

$$X := (\mathbb{R} \times \{\pm 1\}) \,/\, \sim$$

where $(x, 1) \sim (x, -1)$ if $x \neq 0$

$$\forall n \in \mathbb{N}, \quad \left(\frac{1}{2^n}, 1\right) \to \left(\frac{1}{2^{n+1}}, 1\right)$$

$$(1, 1) = (1, -1)$$

$$\downarrow$$

$$\left(\tfrac{1}{2}, 1\right) = \left(\tfrac{1}{2}, -1\right)$$

$$\downarrow$$

$$\left(\tfrac{1}{4}, 1\right) = \left(\tfrac{1}{4}, -1\right)$$

$$(0, 1) \qquad \neq \qquad (0, -1)$$

Line with two origins

$$X := (\mathbb{R} \times \{\pm 1\}) \,/\sim$$

where $(x, 1) \sim (x, -1)$ if $x \neq 0$

$$\forall n \in \mathbb{N}, \quad \left(\frac{1}{2^n}, 1\right) \to \left(\frac{1}{2^{n+1}}, 1\right)$$

$$(1, 1) = (1, -1)$$

$$\downarrow$$

$$\left(\tfrac{1}{2}, 1\right) = \left(\tfrac{1}{2}, -1\right)$$

$$\downarrow$$

$$\left(\tfrac{1}{4}, 1\right) = \left(\tfrac{1}{4}, -1\right)$$

$$\swarrow \qquad \searrow$$

$$(0, 1) \qquad \neq \qquad (0, -1)$$

Cyclic relation

$$X := [0, 2] \subset \mathbb{R}$$

$$\tfrac{1}{2^{n+1}} \rightleftarrows \tfrac{1}{2^n} \qquad 2 - \tfrac{1}{2^n} \rightleftarrows 2 - \tfrac{1}{2^{n+1}}$$

Line with two origins

$$X := (\mathbb{R} \times \{\pm 1\}) / \sim$$

where $(x, 1) \sim (x, -1)$ if $x \neq 0$

$$\forall n \in \mathbb{N}, \quad \left(\frac{1}{2^n}, 1\right) \to \left(\frac{1}{2^{n+1}}, 1\right)$$

$$(1,1) = (1,-1)$$

$$\downarrow$$

$$\left(\tfrac{1}{2}, 1\right) = \left(\tfrac{1}{2}, -1\right)$$

$$\downarrow$$

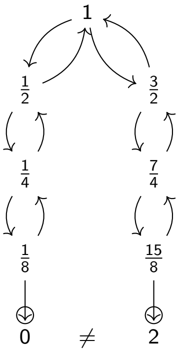
$$\left(\tfrac{1}{4}, 1\right) = \left(\tfrac{1}{4}, -1\right)$$

$$(0,1) \qquad \neq \qquad (0,-1)$$

Cyclic relation

$$X := [0, 2] \subset \mathbb{R}$$

$$\frac{1}{2^{n+1}} \rightleftarrows \frac{1}{2^n} \qquad 2 - \frac{1}{2^n} \rightleftarrows 2 - \frac{1}{2^{n+1}}$$



Third counter-example

$$X := (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$$

where $(\mathbb{N} \cup \{\infty\})$ is endowed with the order topology

Third counter-example

$$X := (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$$

where $(\mathbb{N} \cup \{\infty\})$ is endowed with the order topology

$$\forall n, m \in \mathbb{N}, \quad (n, m) \to (n+1, m) \quad \text{and} \quad (n, m) \to (n, m+1)$$

Note how $(n, m) \xrightarrow{*} (n', m')$ iff $n \leq n'$ and $m \leq m'$

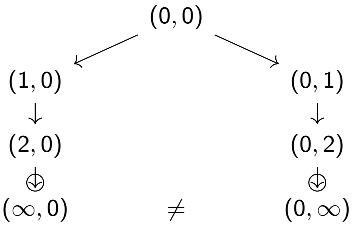
Third counter-example

$$X := (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$$

where $(\mathbb{N} \cup \{\infty\})$ is endowed with the order topology

$$\forall n, m \in \mathbb{N}, \quad (n, m) \to (n+1, m) \quad \text{and} \quad (n, m) \to (n, m+1)$$

Note how $(n, m) \xrightarrow{*} (n', m')$ iff $n \leq n'$ and $m \leq m'$



Valuation

$$\operatorname{val}\left(xy^2z^2 + z^3 + y\right) = 1$$
$$\operatorname{val}\left(x^2yz + xy^2z\right) = 4$$

Valuation

$$\mathrm{val}\left(xy^2z^2 + z^3 + y\right) = 1$$
$$\mathrm{val}\left(x^2yz + xy^2z\right) = 4$$

→

Metric

$$f, g \in \mathbb{K}[x_1, \cdots, x_n]$$
$$\delta(f, g) := \frac{1}{2^{\mathrm{val}(f-g)}}$$

Valuation

$$\mathrm{val}\left(xy^2z^2 + z^3 + y\right) = 1$$
$$\mathrm{val}\left(x^2yz + xy^2z\right) = 4$$

→

Metric

$$f, g \in \mathbb{K}[x_1, \cdots, x_n]$$
$$\delta(f, g) := \frac{1}{2^{\mathrm{val}(f-g)}}$$

↓

Cauchy-completion

Cauchy sequences in $\mathbb{K}[x_1, \cdots, x_n]$ modulo $(x_n)_n \sim (y_n)_n$ if and only if $\lim_{n\to\infty} \delta(x_n, y_n) = 0$

Valuation

$$\operatorname{val}\left(xy^2z^2 + z^3 + y\right) = 1$$
$$\operatorname{val}\left(x^2yz + xy^2z\right) = 4$$

Metric

$$f, g \in \mathbb{K}[x_1, \cdots, x_n]$$
$$\delta(f, g) := \frac{1}{2^{\operatorname{val}(f-g)}}$$

Cauchy-completion

Cauchy sequences in $\mathbb{K}[x_1, \cdots, x_n]$ modulo $(x_n)_n \sim (y_n)_n$ if and only if $\lim_{n\to\infty} \delta(x_n, y_n) = 0$

Formal power series

Denoted by $\mathbb{K}[[x_1, \cdots, x_n]]$

Structure on $\mathbb{K}[[x_1, \cdots, x_n]]$

Algebra operations on $\mathbb{K}[x_1, \cdots, x_n]$ are **continuous**

Hence **extend naturally** on $\mathbb{K}[[x_1, \cdots, x_n]]$

Therefore $\mathbb{K}[[x_1, \cdots, x_n]]$ is a **topological algebra**

Structure on $\mathbb{K}[[x_1, \cdots, x_n]]$

Algebra operations on $\mathbb{K}[x_1, \cdots, x_n]$ are **continuous**

Hence **extend naturally** on $\mathbb{K}[[x_1, \cdots, x_n]]$

Therefore $\mathbb{K}[[x_1, \cdots, x_n]]$ is a **topological algebra**

Formal power series as infinite linear combinations of monomials

$\mathbb{K}[[x_1, \cdots, x_n]]$ isomorphic to the **dual vector space** of $\mathbb{K}[x_1, \cdots, x_n]$

Since the monoid $[x_1, \cdots, x_n]$ of monomials is a basis of $\mathbb{K}[x_1, \cdots, x_n]$

$$f \in \mathbb{K}[[x_1, \cdots, x_n]] \quad \leftrightarrow \quad f : [x_1, \cdots, x_n] \to \mathbb{K} \text{ a map}$$

Denote $\langle f | m \rangle := f(m)$ then $f =: \sum_{m \in [x_1, \cdots, x_n]} \langle f | m \rangle \, m$

Theorem. [Chenavier, Cluzeau, ML, 2024]

Let I be an ideal of commutative formal power series.

Given any f in the topological closure of I, we can compute a cofactor representation of f with respect to a system of generators of I.

In other words, we prove constructively that I is **topologically closed**.

Theorem. [Chenavier, Cluzeau, ML, 2024]

Let R be a set of commutative formal power series and $<$ be a local monomial order that is compatible with the degree.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

II. IDEALS OF FORMAL POWER SERIES

Local monomial orders

➔ Total order **compatible** with monomial multiplication

Local monomial orders

- ➔ Total order **compatible** with monomial multiplication
- ➔ Local: 1 is **maximal**

Local monomial orders

- Total order **compatible** with monomial multiplication
- Local: 1 is **maximal**
- Compatible with the degree if the total degree function on monomials is **decreasing**.

Local monomial orders

- Total order **compatible** with monomial multiplication
- **Local**: 1 is **maximal**
- **Compatible with the degree** if the total degree function on monomials is **decreasing**.

Consequence: if $<$ is a **local** order **compatible with the degree** then

$$\mathrm{val}(f) = \deg(\mathrm{LM}(f))$$

Local monomial orders

- ➔ Total order **compatible** with monomial multiplication
- ➔ **Local**: 1 is **maximal**
- ➔ **Compatible with the degree** if the total degree function on monomials is **decreasing**.

Consequence: if $<$ is a **local** order **compatible with the degree** then

$$\mathrm{val}\,(f) = \deg\,(\mathrm{LM}\,(f))$$

Rewriting on formal power series: **same as multivariate division on polynomials** but with respect to

- ➔ a **local** order $<$ **compatible with the degree**
- ➔ a set R of non-zero formal power series

Standard bases

Defined syntactically like **Gröbner bases** for polynomials *i.e.*
a subset $G \subseteq I$ of an ideal

$$\langle \mathrm{LM}(G) \rangle = \mathrm{LM}(I)$$

(Note: LM is w.r.t. the local order)

Standard bases

Defined syntactically like **Gröbner bases** for polynomials *i.e.* a subset $G \subseteq I$ of an ideal

$$\langle \mathrm{LM}(G) \rangle = \mathrm{LM}(I)$$

(Note: LM is w.r.t. the local order)

Theorem [Chenavier, 2020].
R is a **standard basis** of the ideal it generates for a **local** order $<$ if, and only if, the system induced by R and $<$ is **topologically confluent**.

Counter-example of topological confluence $\Rightarrow$ confluence

Consider again, in $\mathbb{K}[[x, y, z]]$

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$

Counter-example of topological confluence $\Rightarrow$ confluence

Consider again, in $\mathbb{K}[[x, y, z]]$

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$

R is a **standard basis** because

- ➔ $\mathrm{LM}(R) = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and
- ➔ if $f \in I(R)$ then f has no constant coefficient

Counter-example of topological confluence $\Rightarrow$ confluence

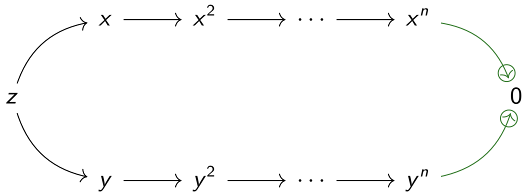
Consider again, in $\mathbb{K}[[x, y, z]]$

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$

R is a **standard basis** because

- ➔ $\mathrm{LM}\,(R) = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and
- ➔ if $f \in I(R)$ then f has no constant coefficient

Thus the system is **topologically confluent**



However we saw previously that it is **not** **confluent**

Ideals of formal power series are topologically closed

- ➔ $\mathbb{K}[[x_1, \cdots, x_n]]$: local noetherian topological ring with respect to the $(x_1, \cdots, x_n)$-adic topology. Therefore a **Zariski ring**
  [Samuel, Zariski, 1975]

Ideals of formal power series are topologically closed

- ➔ $\mathbb{K}[[x_1, \cdots, x_n]]$: local noetherian topological ring with respect to the $(x_1, \cdots, x_n)$-adic topology. Therefore a **Zariski ring** **[Samuel, Zariski, 1975]**
- ➔ Constructive proof providing a **cofactor representation** of a formal power series in the topological closure of the ideal **[Chenavier, Cluzeau, ML, 2024]**

Lemma. Let I be an ideal in $\mathbb{K}[[x_1, \cdots, x_n]]$ and $<$ be a **local** monomial order **compatible with the degree**

Then $\mathrm{LM}\left(\bar{I}\right) = \mathrm{LM}\left(I\right)$ where $\bar{I}$ denotes the **topological closure of I**

Lemma. Let I be an ideal in $\mathbb{K}[[x_1, \cdots, x_n]]$ and $<$ be a **local** monomial order **compatible with the degree**

Then $\mathrm{LM}\left(\bar{I}\right) = \mathrm{LM}\left(I\right)$ where $\bar{I}$ denotes the **topological closure of I**

Proof. If $f \in \bar{I} \setminus \{0\}$ then $\exists (f_k)_k$ in I converging to f
Take f_K such that $\delta(f_K, f) < \frac{1}{2^{\deg(\mathrm{LM}(f))}}$

Lemma. Let I be an ideal in $\mathbb{K}[[x_1, \cdots, x_n]]$ and $<$ be a **local** monomial order **compatible with the degree**

Then $\mathrm{LM}\left(\bar{I}\right) = \mathrm{LM}\left(I\right)$ where $\bar{I}$ denotes the **topological closure of I**

Proof. If $f \in \bar{I} \setminus \{0\}$ then $\exists (f_k)_k$ in I converging to f
Take f_K such that $\delta(f_K, f) < \frac{1}{2^{\deg(\mathrm{LM}(f))}}$

Thus $\deg(\mathrm{LM}\left(f_K - f\right)) > \deg(\mathrm{LM}\left(f\right))$
By **compatibility with the degree** we get $\mathrm{LM}\left(f_K - f\right) < \mathrm{LM}\left(f\right)$
This means that for all $m \geq \mathrm{LM}\left(f\right)$ we have $\langle f_K | m \rangle = \langle f | m \rangle$

Lemma. Let I be an ideal in $\mathbb{K}[[x_1, \cdots, x_n]]$ and $<$ be a **local** monomial order **compatible with the degree**

Then $\mathrm{LM}\left(\bar{I}\right) = \mathrm{LM}(I)$ where $\bar{I}$ denotes the **topological closure of I**

Proof. If $f \in \bar{I} \setminus \{0\}$ then $\exists (f_k)_k$ in I converging to f
Take f_K such that $\delta(f_K, f) < \frac{1}{2^{\deg(\mathrm{LM}(f))}}$

Thus $\deg(\mathrm{LM}(f_K - f)) > \deg(\mathrm{LM}(f))$
By **compatibility with the degree** we get $\mathrm{LM}(f_K - f) < \mathrm{LM}(f)$
This means that for all $m \geq \mathrm{LM}(f)$ we have $\langle f_K | m \rangle = \langle f | m \rangle$

Hence $\mathrm{LM}(f) = \mathrm{LM}(f_K)$ but $f_K \in I$ therefore $\mathrm{LM}(f) \in \mathrm{LM}(I)$

Goal of the proof

Let I be an ideal in $\mathbb{K}[[x_1, \cdots, x_n]]$
Let $<$ a **local** monomial order **compatible with the degree**

Fix $G := \{s_1, \cdots, s_\ell\}$ a **standard basis** of I with respect to $<$

Let f be **in the topological closure** of I
Construct $(f_1, \cdots, f_\ell) \in \mathbb{K}[[x_1, \cdots, x_n]]^\ell$ such that $f = f_1 s_1 + \cdots + f_\ell s_\ell$.

Goal of the proof

Let I be an ideal in $\mathbb{K}[[x_1, \cdots, x_n]]$
Let $<$ a **local** monomial order **compatible with the degree**

Fix $G := \{s_1, \cdots, s_\ell\}$ a **standard basis** of I with respect to $<$

Let f be **in the topological closure** of I
Construct $(f_1, \cdots, f_\ell) \in \mathbb{K}[[x_1, \cdots, x_n]]^\ell$ such that $f = f_1 s_1 + \cdots + f_\ell s_\ell$.

Strategy

Construct a sequence $(f_i^{(k)})_{k \in \mathbb{N}}$ for each $i \in [\![1 \,..\, \ell]\!]$

Prove that they are Cauchy

Take their limits and show that they yield a cofactor representation of f

Consider

$$F_k := f - \sum_{i=1}^{\ell} f_i^{(k)} s_i \in \bar{I}$$

If $F_k = 0$, it's over

Consider

$$F_k := f - \sum_{i=1}^{\ell} f_i^{(k)} s_i \in \overline{I}$$

If $F_k = 0$, it's over

Otherwise, we can eliminate $m_k := \mathrm{LM}(F_k)$ with the **standard basis** G by choosing an $i_k \in [\![1\,..\,\ell]\!]$ and a $q_k \in [x_1, \cdots, x_n]$ such that

$$m_k := \mathrm{LM}(F_k) = q_k \cdot \mathrm{LM}(s_{i_k})$$

Consider

$$F_k := f - \sum_{i=1}^{\ell} f_i^{(k)} s_i \in \overline{I}$$

If $F_k = 0$, it's over

Otherwise, we can eliminate $m_k := \mathrm{LM}\left(F_k\right)$ with the **standard basis** G by choosing an $i_k \in [\![1\,..\,\ell]\!]$ and a $q_k \in [x_1, \cdots, x_n]$ such that

$$m_k := \mathrm{LM}\left(F_k\right) = q_k \cdot \mathrm{LM}\left(s_{i_k}\right)$$

We define $f_i^{(k+1)} := f_i^{(k)}$ for all $i \neq i_k$ and

$$f_{i_k}^{(k+1)} := f_{i_k}^{(k)} + \frac{\mathrm{LC}\left(F_k\right)}{\mathrm{LC}\left(s_{i_k}\right)} q_k$$

Consider

$$F_k := f - \sum_{i=1}^{\ell} f_i^{(k)} s_i \in \bar{I}$$

If $F_k = 0$, it's over

Otherwise, we can eliminate $m_k := \mathrm{LM}(F_k)$ with the **standard basis** G by choosing an $i_k \in [\![1\,..\,\ell]\!]$ and a $q_k \in [x_1, \cdots, x_n]$ such that

$$m_k := \mathrm{LM}(F_k) = q_k \cdot \mathrm{LM}(s_{i_k})$$

We define $f_i^{(k+1)} := f_i^{(k)}$ for all $i \neq i_k$ and

$$f_{i_k}^{(k+1)} := f_{i_k}^{(k)} + \frac{\mathrm{LC}(F_k)}{\mathrm{LC}(s_{i_k})} q_k$$

By induction hypothesis it follows that:

$$F_{k+1} := \sum_{i=1}^{\ell} f_i^{(k+1)} s_i \in \bar{I}$$

Facts

Following from the facts that:

- we have **finitely many** variables
- $<$ is **compatible with the degree**
- the sequence $(m_k)_k$ of eliminated monomials is strictly decreasing

Facts

Following from the facts that:

- we have **finitely many** variables
- $<$ is **compatible with the degree**
- the sequence $(m_k)_k$ of eliminated monomials is strictly decreasing

we have that the sequences $(f_i^{(k)})_k$ are Cauchy for any $i \in [\![1 \mathinner{..} \ell]\!]$

Facts

Following from the facts that:

- we have **finitely many** variables
- $<$ is **compatible with the degree**
- the sequence $(m_k)_k$ of eliminated monomials is strictly decreasing

we have that the sequences $(f_i^{(k)})_k$ are Cauchy for any $i \in [\![1\,..\,\ell]\!]$

Denote by $f_i^{(\infty)}$ their respective limits
Then by continuity of the metric and the facts above it follows that:

$$\delta\left(f - \sum_{i=1}^{\ell} f_i^{(\infty)} s_i, 0\right) = 0$$

hence our desired result

III. APPLICATIONS

Proposition. For all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f \xrightarrow{\circledast} g$ then $f - g \in I$

Proposition. For all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f \mathrel{-\!\!\circledcirc} g$ then $f - g \in I$

Proof. $f \mathrel{-\!\!\circledcirc} g$ implies the existence of a sequence $f_k \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \xrightarrow{*} f_k$ and $\delta(f_k, g) < 2^{-k}$ so that $\lim_{k\to\infty} f_k = g$

Proposition. For all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f \to_{\mathcal{T}} g$ then $f - g \in I$

Proof. $f \to_{\mathcal{T}} g$ implies the existence of a sequence $f_k \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \xrightarrow{*} f_k$ and $\delta(f_k, g) < 2^{-k}$ so that $\lim_{k \to \infty} f_k = g$

By the same reasoning as polynomial reduction, $f \xrightarrow{*} f_k$ implies $f - f_k \in I$ thus at the limit we obtain $\lim_{k \to \infty}(f - f_k) = f - g \in \overline{I}$

Proposition. For all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f \rightarrow\!\!\!\!\!\!\bigcirc\!\!\!\!\!\!\!\!\;\; g$ then $f - g \in I$

Proof. $f \rightarrow\!\!\!\!\!\!\bigcirc\;\; g$ implies the existence of a sequence $f_k \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \xrightarrow{*} f_k$ and $\delta(f_k, g) < 2^{-k}$ so that $\lim_{k \to \infty} f_k = g$

By the same reasoning as polynomial reduction, $f \xrightarrow{*} f_k$ implies $f - f_k \in I$ thus at the limit we obtain $\lim_{k \to \infty}(f - f_k) = f - g \in \overline{I}$

But I is topologically closed, hence $f - g \in I$

Theorem. [Chenavier, Cluzeau, ML, 2024]

Let R be a set of formal power series and $<$ be a **local** monomial order that is **compatible with the degree**.

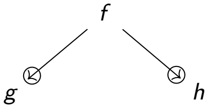
The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

Theorem. [Chenavier, Cluzeau, ML, 2024]

Let R be a set of formal power series and $<$ be a **local** monomial order that is **compatible with the degree**.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

Strategy: Given

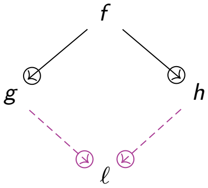


Theorem. [Chenavier, Cluzeau, ML, 2024]

Let R be a set of formal power series and $<$ be a **local** monomial order that is **compatible with the degree**.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

Strategy: Given



Close the diagram

- Fix R a non-empty set of non-zero formal power series
- Fix $<$ a **local** monomial order **compatible with the degree**
- Write $\rightarrow$ the one-step rewriting relation induced by R and $<$

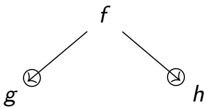
- Fix R a non-empty set of non-zero formal power series
- Fix $<$ a **local** monomial order **compatible with the degree**
- Write $\to$ the one-step rewriting relation induced by R and $<$

Assume that $\to$ is **topologically confluent** *i.e.* R is a **standard basis** with respect to $<$ of the ideal $I := I(R)$ generated by R

- Fix R a non-empty set of non-zero formal power series
- Fix $<$ a **local** monomial order **compatible with the degree**
- Write $\rightarrow$ the one-step rewriting relation induced by R and $<$

Assume that $\rightarrow$ is **topologically confluent** *i.e.* R is a **standard basis** with respect to $<$ of the ideal $I := I(R)$ generated by R

Let $f, g, h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that:

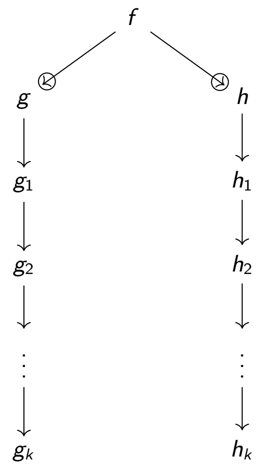


Goal

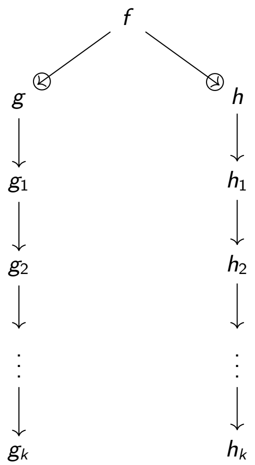
Construct inductively **two rewriting sequences** starting from g and h respectively that will be proven to be **Cauchy**

It will turn out that the limits are then equal and hence give a **common topological successor** to g and h

➔ By induction:
$\exists g \xrightarrow{*} g_k$ and $\exists h \xrightarrow{*} h_k$

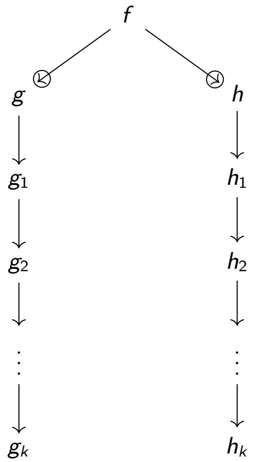


- By induction:
  $\exists g \xrightarrow{*} g_k$ and $\exists h \xrightarrow{*} h_k$
- If $g_k = h_k$, then it's over!



- By induction:
  $\exists g \xrightarrow{*} g_k$ and $\exists h \xrightarrow{*} h_k$
- If $g_k = h_k$, then it's over!
- From the previous proposition:

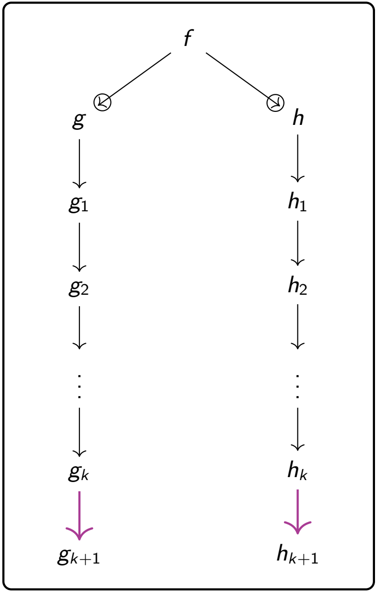
$$g_k - h_k \in I$$



- ➔ By induction:
  $\exists g \xrightarrow{*} g_k$ and $\exists h \xrightarrow{*} h_k$
- ➔ If $g_k = h_k$, then it's over!
- ➔ From the previous proposition:

$$g_k - h_k \in I$$

- ➔ Rewrite LM $(g_k - h_k)$



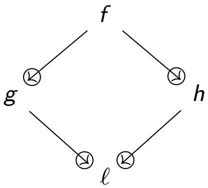
Facts

- ➔ the sequences $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ are **Cauchy**
- ➔ their limits are **equal**

Facts

- ➔ the sequences $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ are **Cauchy**
- ➔ their limits are **equal**

So $\lim_{k\to\infty} g_k = \lim_{k\to\infty} h_k =: \ell$



Which shows that $\to$ is **infinitary confluent**

Let R be a set of non-zero formal power series.
Let $<$ be a **local** monomial order **compatible with the degree**.
Write $\rightarrow$ the relation induced by R and $<$ and $\xrightarrow{\tau}$ the topological rewriting relation associated to it.

Let R be a set of non-zero formal power series.
Let $<$ be a **local** monomial order **compatible with the degree**.
Write $\to$ the relation induced by R and $<$ and $\overset{\infty}{\to}$ the topological rewriting relation associated to it.

Then the following properties are equivalent:

(i) the system is **topologically confluent**

(ii) for all $f \in I$, we have $f \overset{\infty}{\to} 0$

(iii) for all $f \in I \setminus \{0\}$, we have f reducible

(iv) for all $f \in I \setminus \{0\}$, we have $\mathrm{LM}(f)$ reducible

(v) R is a **standard basis**

(vi) the set of normal forms forms a canonical set of representatives for the quotient algebra $\mathbb{K}[[x_1, \cdots, x_n]]$ modulo $I(R)$

IV. CONCLUSION AND PERSPECTIVES

Conclusion and perspectives

Summary of presented notions and results:

- we introduced the basic ideas of **topological rewriting theory**
- we proved constructively that ideals of commutative formal power series are **topologically closed**
- we showed that **topological confluence** is equivalent to **infinitary confluence** for formal power series

Further works:

- show that the topological rewriting relation induces convergent rewriting chains in the context of formal power series
- adapt Newman's lemma to topological rewriting theory
- develop computational tools to study Taylor series and formal solutions of PDEs

Conclusion and perspectives

Summary of presented notions and results:

- we introduced the basic ideas of **topological rewriting theory**
- we proved constructively that ideals of commutative formal power series are **topologically closed**
- we showed that **topological confluence** is equivalent to **infinitary confluence** for formal power series

Further works:

- show that the topological rewriting relation induces convergent rewriting chains in the context of formal power series
- adapt Newman's lemma to topological rewriting theory
- develop computational tools to study Taylor series and formal solutions of PDEs

THANK YOU FOR LISTENING!