Topological rewriting on formal power series

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1 Introduction

Following the preliminary work done in [1] and in our latest journal submission [2], we investigate the computational aspects on algebras of formal power series through the lens of standard bases and topological rewriting theory. Just as polynomial reduction strikes in similarity with first-order term rewriting, the study of rewriting on formal power series is analogous in many ways to infinitary term rewriting as well as infinitary λ-calculus. However, the confluence properties naturally arising in each field are syntactically different: namely *topological confluence* and *infinitary confluence*. In [2] we provide examples demonstrating that, in the setting of arbitrary topological rewriting systems, infinitary confluence is in general stronger than topological confluence, but we prove that in the context of formal power series, the two notions are actually equivalent. Many questions are still left unanswered; we present some of them here and provide partial results.

2 Topological rewriting systems

Definition 1 (Topological rewriting system). *A* topological rewriting system *consists of a topological space (X, τ) and a binary relation $\to$ on X.*

We denote by $\xrightarrow{\star}$ the reflexive transitive closure of $\to$ and by $\multimap\!\!\!\circledast$ the topological closure of $\xrightarrow{\star}$ for the product topology $\tau_{\mathrm{dis}}^X \times \tau$, where τ_{dis}^X means the discrete topology on X.

We say that x *rewrites finitely into* y if $x \xrightarrow{\star} y$ and that x *rewrites topologically into* y if $x \multimap\!\!\!\circledast y$. We have the following characterisation of the so-called *topological rewriting relation* $\multimap\!\!\!\circledast$: for all $x, y \in X$, we have $x \multimap\!\!\!\circledast y$ if and only if, for all neighbourhood U of y for the topology τ, there exists $z \in U$ such that $x \xrightarrow{\star} z$.

We recall the definition of normal forms and extend the meaning to our setting. We say that $n \in X$ is a *normal form* if there exists no $x \in X$ such that we have $n \to x$. Let $a \in X$ and $n \in X$ a normal form. We say that n is *a discrete* (resp. *topological*) *normal form of* a if $a \xrightarrow{\star} n$ (resp. $a \multimap\!\!\!\circledast n$).

We now give definitions of three notions of confluence of interest to us.

Definition 2 (Confluences). *Let $(X, \tau, \to)$ be a topological rewriting system.*

*(i) The system is (*finitary*)* confluent *if, for all a that rewrites finitely into b and c, there exists d such that b and c rewrite finitely into d.*
(ii) The system is topologically confluent *if, for all a that rewrites finitely into b and c, there exists d such that b and c rewrite topologically into d.*
(iii) The system is infinitary confluent *if, for all a that rewrites topologically into b and c, there exists d such that b and c rewrite topologically into d.*

Remark 1 (Diagrams of confluences).

(i) Finitary

$$\begin{tikzcd}[column sep=small] & a \arrow[dl, "\star"'] \arrow[dr, "\star"] & \\ b \arrow[dr, dashed, "\star"'] & & c \arrow[dl, dashed, "\star"] \\ & d & \end{tikzcd}$$

(ii) Topological

$$\begin{tikzcd}[column sep=small] & a \arrow[dl, "\star"'] \arrow[dr, "\star"] & \\ b \arrow[dr, dashed, multimap] & & c \arrow[dl, dashed, multimap] \\ & d & \end{tikzcd}$$

(iii) Infinitary

$$\begin{tikzcd}[column sep=small] & a \arrow[dl, multimap] \arrow[dr, multimap] & \\ b \arrow[dr, dashed, multimap] & & c \arrow[dl, dashed, multimap] \\ & d & \end{tikzcd}$$

From the definitions of the relations, we see that $x \multimap y$ follows from $x \xrightarrow{\star} y$. Therefore, we deduce that both finitary and infinitary confluences imply topological confluence. In the next section, we will give a counter-example for the statement “topological confluence implies finitary confluence”. To give a counter-example of the claim “topological confluence implies infinitary confluence”, we must first notice that infinitary confluence is syntactically equivalent to the diamond property for the classical abstract rewriting system $(X, \multimap)$. But, it can be shown by induction that the diamond property of $\multimap$ entails the finitary confluence of $\multimap$ which in turn means that the unique normal form property is verified. Hence we have the following result.

Proposition 1. *Let $(X, \tau, \to)$ be a topological rewriting system. Assume that the system is infinitary confluent. Then, let $a \in X$ and $n, n' \in X$ be topological normal forms of a, we have $n = n'$.*

We now are able to provide our first counter-example.

Example 1 (Contradicting “topological confluence $\Rightarrow$ infinitary confluence”). We present [2, Example 4.1.4]. Consider $X := (\mathbb{N} \cup \{\infty\})^2$ endowed with τ the product topology of the order topology and the relation $\to$ defined by:

$$\forall n, m \in \mathbb{N}, \qquad (n, m) \to (n+1, m) \qquad \text{and} \qquad (n, m) \to (n, m+1).$$

This relation is finitary confluent and hence topologically confluent.

However, notice how, for $n \in \mathbb{N}$ such that $n \geq 1$:

$$(\infty, 0) \multimapinv (n, 0) \longleftarrow \cdots \longleftarrow (1, 0) \longleftarrow (0, 0) \longrightarrow (0, 1) \longrightarrow \cdots \longrightarrow (0, n) \multimap (0, \infty)$$

Therefore, $(\infty, 0)$ and $(0, \infty)$ are distinct topological normal forms of $(0, 0)$. Hence, the system cannot be infinitary confluent.

3 Rewriting on formal power series

We denote by $\mathbb{K}[[x_1, \cdots, x_n]]$ the topological algebra of formal power series in n indeterminates over a field $\mathbb{K}$. We consider the opposite order $<$ of the deglex induced by an arbitrary total order on $\{x_1, \cdots, x_n\}$. Other semigroup orders would work as long as 1 is maximal and the degree function is decreasing. For any non-zero formal power series f, we denote by $\mathrm{lm}(f)$ (resp. by $\mathrm{lc}(f)$) the largest monomial for $<$ appearing in f called the *leading monomial* (resp. the coefficient of the leading monomial called the *leading coefficient*) of f and finally we write the remainder $r(f) := \mathrm{lc}(f)\,\mathrm{lm}(f) - f$. The $(x_1, \cdots, x_n)$-adic topology denoted τ_δ is actually induced by the following metric on $\mathbb{K}[[x_1, \cdots, x_n]]$:

$$\delta(f, g) := \frac{1}{2^{\deg(\mathrm{lm}(f-g))}}.$$

Fix R a set of non-zero formal power series and write I the ideal it generates in $\mathbb{K}[[x_1, \cdots, x_n]]$. Define the reduction relation on formal power series just as in polynomial reduction where we replace a multiple $m\cdot\mathrm{lm}(s)$ of a leading monomial of a rule $s \in R$ with non-zero coefficient by the the remainder $\frac{1}{\mathrm{lc}(s)} m \times r(s)$.

There exists a definition in the context of formal power series that is syntactically analoguous to Gröbner bases called *standard bases* originally introduced in [4]. (See [3] for a modern introduction). The set R of non-zero formal power series is called a *standard basis* of the ideal I if it is finite and for all $f \in I$ there exist $s \in R$ and a monomial m such that $\mathrm{lm}(f) = m \cdot \mathrm{lm}(s)$.

We recall [1, Theorem 4.1.3] as follows: the set R is a standard basis of I if, and only if, the system $(\mathbb{K}[[x_1, \cdots, x_n]], \tau_\delta, \to)$ is topologically confluent.

We can now provide a counter-example to a previous converse implication.

Example 2 (Contradicting "topological confluence $\Rightarrow$ finitary confluence").
We present [1, Example 4.1.4]. Consider $R := \{z-x, z-y, x-x^2, y-y^2\}$ and $<$ the inverse deglex order such that $z > y > x$. We see that R is a standard basis, hence the system is topologically confluent. However, consider the branching:

$$\begin{tikzcd} & x \arrow[r] & x^2 \arrow[r] & x^3 \arrow[r] & \cdots \arrow[r] & x^n \\ z \arrow[ur, bend left] \arrow[dr, bend right] & & & & & \\ & y \arrow[r] & y^2 \arrow[r] & y^3 \arrow[r] & \cdots \arrow[r] & y^n \end{tikzcd}$$

The two branches will never have a common element in a finite amount of steps. Hence the system is not finitary confluent.

Using the fact (known from [5] and proved constructively in [2]) that ideals of formal power series are topologically closed, we obtain the following result:

Theorem 1 ([2, Theorem 4.2.2]). *The system $(\mathbb{K}[[x_1, \cdots, x_n]], \tau_\delta, \to)$ is infinitary confluent if and only if it is topologically confluent.*

4 Perspectives

The definition of the relation $\multimap$ does not explicitly guarantee the existence of rewriting chains. In fact, here is an example showing that it is possible for a to rewrite topologically into b but no rewriting chains of even infinite length exist from a to b.

Example 3. Consider, in $\mathbb{K}[[x, y]]$, the system induced by $R := \{x - y^n | n \geq 1\}$ and the inverse deglex order such that $x > y$. Then we have $x \multimap 0$ but no rewriting chains (*i.e.* $\xrightarrow{\star}$-increasing sequences) starting from x converges to 0.

It still remains to prove or disprove the following conjecture.

Conjecture 1. For the theory of rewriting on formal power series, if the set of rules R is finite, then $f \multimap g$ implies the existence of a rewriting chain starting from f converging to g. An equivalent formulation is as follows: if R is finite, then for all $f \neq g$ such that $f \multimap g$, there exists a direct successor f' of f such that $f \to f' \multimap g$.

We proved the particular case where f has only finitely many direct successors (for instance, if it is a polynomial), as well as the case where R is a standard basis and g is a normal form.

If that conjecture is true, then this actually yields an alternative proof for the fact that, if R is finite, topological confluence of the system induced by R is equivalent to R being a standard basis. Moreover, it would also prove that the transitive closure of the symmetric closure of $\multimap$ is exactly the congruence relation modulo the ideal I generated by R.

References

1. Chenavier, C.: Topological rewriting systems applied to standard bases and syntactic algebras. J. Algebra **550**, 410–431 (2020). https://doi.org/10.1016/j.jalgebra.2019.12.007
2. Chenavier, C., Cluzeau, T., Musson-Leymarie, A.: Topological closure of formal power series ideals and application to topological rewriting theory. Preprint, arXiv:2402.05511 [math.AC] (2024) (2024)
3. Greuel, G.M., Pfister, G.: A Singular introduction to commutative algebra. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann. Berlin: Springer (2002)
4. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. Ann. of Math. (2) **79**, 109–203; **79** (1964), 205–326 (1964). https://doi.org/10.2307/1970547
5. Zariski, O., Samuel, P.: Commutative algebra. Vol. II, Graduate Texts in Mathematics, vol. Vol. 29. Springer-Verlag, New York-Heidelberg (1975), reprint of the 1960 edition