Topological rewriting systems: confluences and chains

Adya Musson-Leymarie

https://adyaml.com

with **Cyrille Chenavier** and **Thomas Cluzeau**

CASC - Computer Algebra for Scientific Computing

September 2nd, 2024

Université de Limoges

Institut de Recherche xlim

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

Instances

In computer science

- ➔ Term rewriting
- ➔ β-reduction in λ-calculus

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

Instances

In computer science

- Term rewriting
- β-reduction in λ-calculus

In computer algebra

- Polynomial reduction
- Involutive divisions

Rewriting theory

Describes sequences of **computations** through **oriented identities** *a.k.a.* **rewrite rules**

Instances

In computer science

- ➔ Term rewriting
- ➔ β-reduction in λ-calculus

In computer algebra

- ➔ Polynomial reduction
- ➔ Involutive divisions

Abstraction

Abstract rewriting theory

Abstract properties common to all concrete rewriting systems: **termination**, **confluence**, **normal forms**

Abstract Rewriting System

- A an underlying set
- $\rightarrow$ a binary relation on A

We write $a \rightarrow b$ for $(a, b) \in \rightarrow$

Abstract Rewriting System

- A an underlying set
- $\rightarrow$ a binary relation on A

We write $a \rightarrow b$ for $(a, b) \in \rightarrow$

Transitive reflexive closure

We write $a \overset{*}{\rightarrow} b$ to express that $a = a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_\ell = b$

Abstract Rewriting System

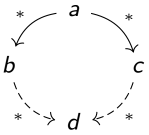
- ➔ A an underlying set
- ➔ $\to$ a binary relation on A

We write $a \to b$ for $(a, b) \in \to$

Transitive reflexive closure

We write $a \xrightarrow{*} b$ to express that $a = a_0 \to a_1 \to \cdots \to a_\ell = b$

Confluence



Abstract Rewriting System

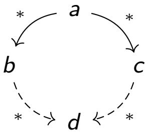
- A an underlying set
- $\rightarrow$ a binary relation on A

We write $a \rightarrow b$ for $(a, b) \in \rightarrow$

Transitive reflexive closure

We write $a \xrightarrow{*} b$ to express that $a = a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_\ell = b$

Confluence



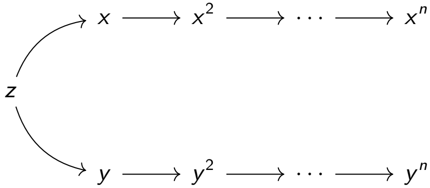
Example

Multivariate division with respect to R is **confluent** iff R is a **Gröbner basis**

Confluence "at the limit"

In $\mathbb{K}[[x, y, z]]$ with the inverse deglex order such that $z > y > x$ take

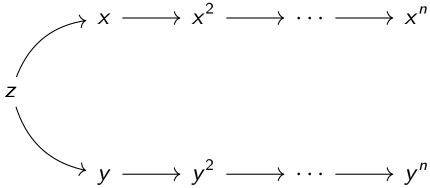
$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$



Confluence "at the limit"

In $\mathbb{K}[[x, y, z]]$ with the inverse deglex order such that $z > y > x$ take

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$



The two branches will never have a common element
Hence the system is **not** **confluent**

However with the (x, y, z)-adic topology both branches converge to 0

Topological Abstract Rewriting System

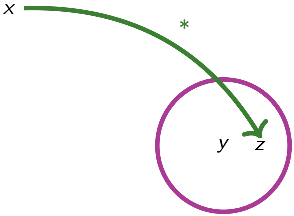
- (X, τ) a **topological space**
- $\to$ a binary relation on X

Topological Abstract Rewriting System

- (X, τ) a **topological space**
- $\to$ a binary relation on X

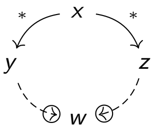
Topological rewriting relation

Write $x \to\!\!\!\!\!\!\bigcirc\; y$ if for **every neighbourhood** U of y **there exists** $z \in U$ s.t. $x \xrightarrow{*} z$

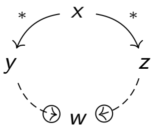


Note how $x \xrightarrow{*} y$ implies $x \to\!\!\!\!\!\!\bigcirc\; y$

Topological confluence

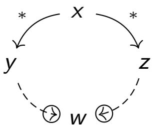


Topological confluence

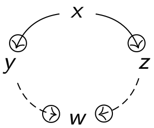


Theorem [Chenavier 2020].
Standard basis ⇔ topological confluence
where **standard bases** are to formal power series as **Gröbner bases** are to polynomials

Topological confluence



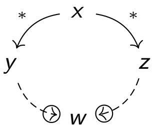
Infinitary confluence



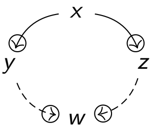
Theorem [Chenavier 2020].
Standard basis ⇔ **topological confluence**
where **standard bases** are to formal power series as **Gröbner bases** are to polynomials

Of interest in computer science: **infinitary λ/Σ-terms**

Topological confluence



Infinitary confluence



Theorem [Chenavier 2020].
Standard basis $\Leftrightarrow$ topological confluence
where **standard bases** are to formal power series as **Gröbner bases** are to polynomials

Of interest in computer science: **infinitary λ/Σ-terms**

It implies the **Church-Rosser property** of the classical system $(X, \rightarrow)$
Under certain topological separation conditions, it implies **unicity of normal forms** reached by $\rightarrow$

Strength of confluences

For every TARS we have:

confluence $\Longrightarrow$ **topological confluence**

infinitary confluence $\Longrightarrow$ **topological confluence**

Strength of confluences

For every TARS we have:
confluence $\Longrightarrow$ **topological confluence**
infinitary confluence $\Longrightarrow$ **topological confluence**

Discrete rewriting system

If $x \twoheadrightarrow\!\!\!\!\!\!\bigcirc\ y$ implies $x \xrightarrow{*} y$, then we say that the TARS $(X, \tau, \to)$ has **discrete rewriting**

In such a case, **confluence**, **topological confluence** and **infinitary confluence** are **trivially equivalent**

For instance, if τ is the **discrete topology**, then $(X, \tau, \to)$ has **discrete rewriting**

Counter-example of topological confluence $\Rightarrow$ confluence

Consider again, in $\mathbb{K}[[x, y, z]]$

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$

Counter-example of topological confluence $\Rightarrow$ confluence

Consider again, in $\mathbb{K}[[x, y, z]]$

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$

R is a **standard basis** because

- ➔ $\mathrm{LM}(R) = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and
- ➔ if $f \in I(R)$ then f has no constant coefficient

Counter-example of topological confluence $\Rightarrow$ confluence

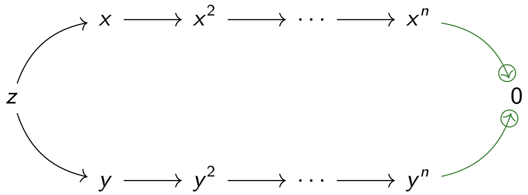
Consider again, in $\mathbb{K}[[x, y, z]]$

$$R = \{\mathbf{z} - y, \quad \mathbf{z} - x, \quad \mathbf{y} - y^2, \quad \mathbf{x} - x^2\}.$$

R is a **standard basis** because

- ➔ $\operatorname{LM}(R) = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and
- ➔ if $f \in I(R)$ then f has no constant coefficient

Thus the system is **topologically confluent**



However we saw previously that it is **not** **confluent**

Counter-example of topological confluence $\Rightarrow$ infinitary confluence

$$X := (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$$

where $(\mathbb{N} \cup \{\infty\})$ is endowed with the order topology

Counter-example of topological confluence $\Rightarrow$ infinitary confluence

$$X := (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$$

where $(\mathbb{N} \cup \{\infty\})$ is endowed with the order topology

$$\forall n, m \in \mathbb{N}, \quad (n, m) \to (n+1, m) \quad \text{and} \quad (n, m) \to (n, m+1)$$

Note how $(n, m) \xrightarrow{*} (n', m')$ iff $n \leq n'$ and $m \leq m'$

Counter-example of topological confluence $\Rightarrow$ infinitary confluence

$$X := (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$$

where $(\mathbb{N} \cup \{\infty\})$ is endowed with the order topology

$$\forall n, m \in \mathbb{N}, \quad (n, m) \to (n + 1, m) \quad \text{and} \quad (n, m) \to (n, m + 1)$$

Note how $(n, m) \xrightarrow{*} (n', m')$ iff $n \leq n'$ and $m \leq m'$

$$\begin{array}{ccc} & (0,0) & \\ \swarrow & & \searrow \\ (1,0) & & (0,1) \\ \downarrow & & \downarrow \\ (2,0) & & (0,2) \\ \circledcirc & & \circledcirc \\ (\infty, 0) & \neq & (0, \infty) \end{array}$$

Theorem [Chenavier, Cluzeau, ML, 2024] using [Chenavier, 2020].

Let R be a set of formal power series and $<$ be a local monomial order that is compatible with the degree.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

Theorem [Chenavier, Cluzeau, ML, 2024] using [Chenavier, 2020].

Let R be a set of formal power series and $<$ be a local monomial order that is compatible with the degree.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

Alternatively

Find conditions on topological rewriting systems, in particular verified by formal power series, that are sufficient to prove

topological confluence equivalent to **infinitary confluent**

and then prove that it implies

topological confluence equivalent to **standard bases**
for commutative formal power series

(See *Notes on topological rewriting theory* at adyaml.com/research)

Valuation

$$\operatorname{val}\left(xy^2z^2 + z^3 + y\right) = 1$$
$$\operatorname{val}\left(x^2yz + xy^2z\right) = 4$$

→

Metric

$$f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$$
$$\delta(f, g) := \frac{1}{2^{\operatorname{val}(f-g)}}$$

Valuation

$$\operatorname{val}\left(xy^2z^2 + z^3 + y\right) = 1$$
$$\operatorname{val}\left(x^2yz + xy^2z\right) = 4$$

Metric

$$f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$$
$$\delta(f, g) := \frac{1}{2^{\operatorname{val}(f-g)}}$$

Example of a convergent sequence

In $\mathbb{K}[[x, y, z]]$ the sequence (f_n) of powers of a variable (say x) converges: $\lim_{n\to\infty} f_n = 0$ because $\operatorname{val}(x^n - 0) \underset{n\to\infty}{\longrightarrow} \infty$

Valuation

$$\mathrm{val}\left(xy^2z^2 + z^3 + y\right) = 1$$

$$\mathrm{val}\left(x^2yz + xy^2z\right) = 4$$

Metric

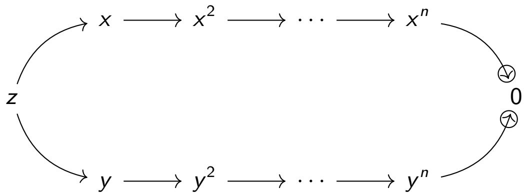
$$f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$$

$$\delta(f, g) := \frac{1}{2^{\mathrm{val}(f-g)}}$$

Example of a convergent sequence

In $\mathbb{K}[[x, y, z]]$ the sequence (f_n) of powers of a variable (say x) converges: $\lim_{n\to\infty} f_n = 0$ because $\mathrm{val}\left(x^n - 0\right) \underset{n\to\infty}{\longrightarrow} \infty$

Hence in the example of the introduction:



Monomial orders

- Total order compatible with monomial multiplication
- **Global** if 1 is minimal $\rightarrow$ **Gröbner bases**
- **Local** if 1 is maximal $\rightarrow$ **Standard bases**
- **Compatible with the degree** if the degree function on monomials is non-increasing (resp. non-decreasing) for a **local** (resp. **global**) order

Consequence: if $<$ is a **local** order **compatible with the degree** then

$$\mathrm{val}(f) = \deg(\mathrm{LM}(f))$$

Ideals of formal power series are topologically closed

➔ $\mathbb{K}[[x_1, \cdots, x_n]]$: local noetherian topological ring with respect to the $(x_1, \cdots, x_n)$-adic topology. Therefore a **Zariski ring** **[Samuel, Zariski, 1975]**

Ideals of formal power series are topologically closed

- $\mathbb{K}[[x_1, \cdots, x_n]]$: local noetherian topological ring with respect to the $(x_1, \cdots, x_n)$-adic topology. Therefore a **Zariski ring** **[Samuel, Zariski, 1975]**
- Constructive proof providing a **cofactor representation** of a formal power series in the topological closure of the ideal **[Chenavier, Cluzeau, ML, 2024]**

Theorem [Chenavier, Cluzeau, ML, 2024].

Let R be a set of formal power series and $<$ be a **local** monomial order that is **compatible with the degree**.

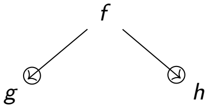
The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

Theorem [Chenavier, Cluzeau, ML, 2024].

Let R be a set of formal power series and $<$ be a **local** monomial order that is **compatible with the degree**.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

Strategy: Given

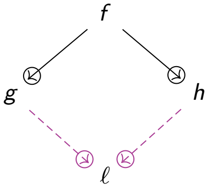


Theorem [Chenavier, Cluzeau, ML, 2024].

Let R be a set of formal power series and $<$ be a **local** monomial order that is **compatible with the degree**.

The rewriting system induced by R and $<$ is **topologically confluent** if and only if it is **infinitary confluent**.

Strategy: Given



Close the diagram

- Fix R a non-empty set of non-zero formal power series
- Fix $<$ a **local** monomial order **compatible with the degree**
- Write $\rightarrow$ the one-step rewriting relation induced by R and $<$

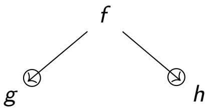
- ➔ Fix R a non-empty set of non-zero formal power series
- ➔ Fix $<$ a **local** monomial order **compatible with the degree**
- ➔ Write $\rightarrow$ the one-step rewriting relation induced by R and $<$

Assume that $\rightarrow$ is **topologically confluent** *i.e.* R is a **standard basis** with respect to $<$ of the ideal $I := I(R)$ generated by R

- Fix R a non-empty set of non-zero formal power series
- Fix $<$ a **local** monomial order **compatible with the degree**
- Write $\rightarrow$ the one-step rewriting relation induced by R and $<$

Assume that $\rightarrow$ is **topologically confluent** *i.e.* R is a **standard basis** with respect to $<$ of the ideal $I := I(R)$ generated by R

Let $f, g, h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that:

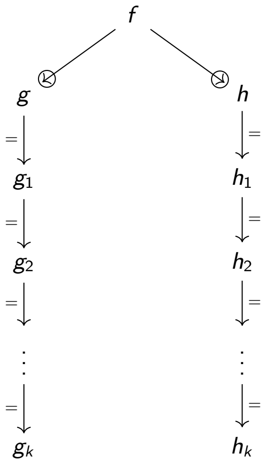


Goal

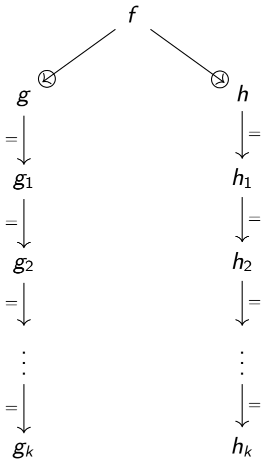
Construct inductively **two rewriting sequences** starting from g and h respectively that will be proven to be **Cauchy**

It will turn out that the limits are then equal and hence give a **common topological successor** to g and h

- By induction:
  $\exists g \xrightarrow{*} g_k$ and $\exists h \xrightarrow{*} h_k$

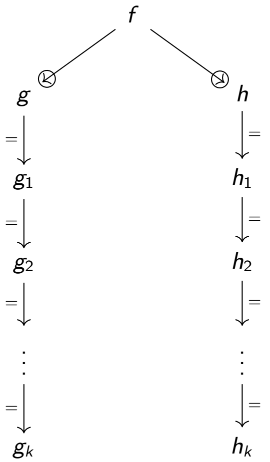


- ➔ By induction:
  $\exists g \xrightarrow{*} g_k$ and $\exists h \xrightarrow{*} h_k$
- ➔ If $g_k = h_k$, then it's over!



- By induction:
  $\exists g \xrightarrow{*} g_k$ and $\exists h \xrightarrow{*} h_k$
- If $g_k = h_k$, then it's over!
- Otherwise:

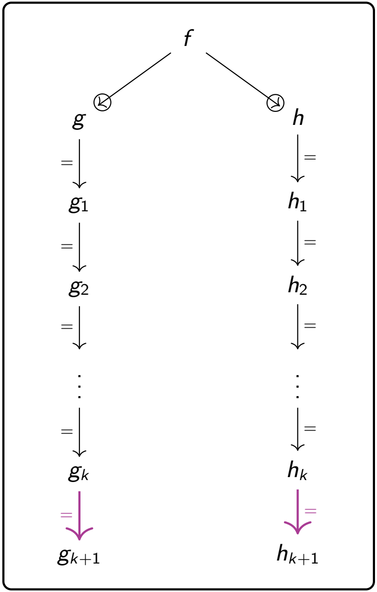
$$0 \neq g_k - h_k \in I$$



- By induction:
  $\exists g \overset{*}{\to} g_k$ and $\exists h \overset{*}{\to} h_k$
- If $g_k = h_k$, then it's over!
- Otherwise:

$$0 \neq g_k - h_k \in I$$

- Rewrite LM $(g_k - h_k)$



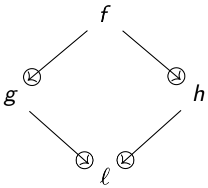
Facts

- ➔ the sequences $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ are **Cauchy**
- ➔ their limits are **equal**

Facts

- ➔ the sequences $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ are **Cauchy**
- ➔ their limits are **equal**

So $\lim_{k\to\infty} g_k = \lim_{k\to\infty} h_k =: \ell$

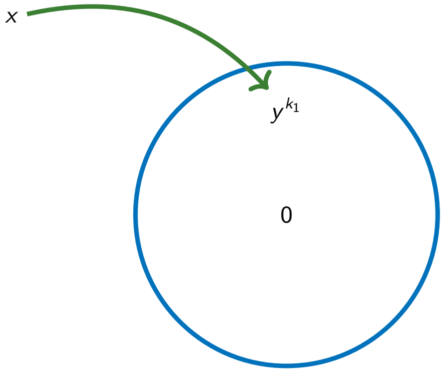


Which shows that $\to$ is **infinitary confluent**

Problem. $a \xrightarrow{\circ} b$ does not always imply $a \to a_1 \to a_2 \to a_3 \to \cdots \underset{\infty}{\longrightarrow} b$

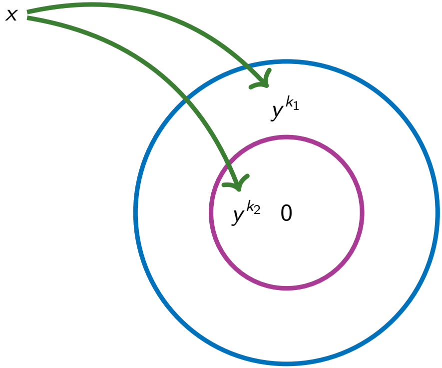
Problem. $a \xrightarrow{\circledast} b$ does not always imply $a \to a_1 \to a_2 \to a_3 \to \cdots \underset{\infty}{\longrightarrow} b$

Consider in $\mathbb{K}[[x, y]]$ the rules $R := \{x - y^k | k \geq 1\}$ with local deglex $x > y$



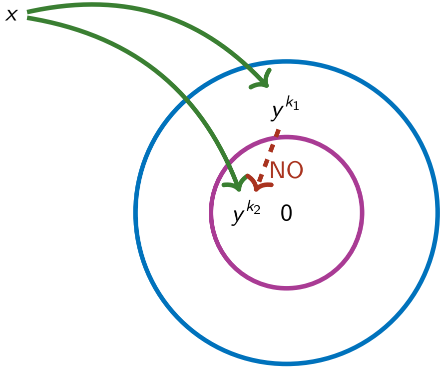
Problem. $a \xrightarrow{\circledast} b$ does not always imply $a \to a_1 \to a_2 \to a_3 \to \cdots \underset{\infty}{\longrightarrow} b$

Consider in $\mathbb{K}[[x, y]]$ the rules $R := \{x - y^k | k \geq 1\}$ with local deglex $x > y$



Problem. $a \xrightarrow{\circledast} b$ does not always imply $a \to a_1 \to a_2 \to a_3 \to \cdots \underset{\infty}{\longrightarrow} b$

Consider in $\mathbb{K}[[x,y]]$ the rules $R := \{x - y^k | k \geq 1\}$ with local deglex $x > y$



Conjecture (Chains).
Consider commutative formal power series with a **finite number of rules**. If $f \xrightarrow{\circledast} \alpha$, with α a normal form, then there exists a (possibly infinite) rewriting chain from f to α

Conjecture (Chains).
Consider commutative formal power series with a **finite number of rules**. If $f \xrightarrow{\circledast} \alpha$, with α a normal form, then there exists a (possibly infinite) rewriting chain from f to α

Has been proven when rules form a **standard basis** or when f is a series with only finitely many multiple of leading monomials of rules (in particular, if f **is a polynomial**)

Conclusion and perspectives

Summary of presented notions and results:

- we introduced different confluence properties for topological rewriting systems
- we provided counter-examples for converse strength implications
- we showed that **topological confluence** is equivalent to **infinitary confluence** for formal power series thanks to the topological closure of ideals

Further works:

- prove or disprove the **chains conjecture**
- find a generalised Newman's lemma in the topological setting
- develop computational tools to manipulate Taylor series to ultimately investigate formal solutions of PDEs

Conclusion and perspectives

Summary of presented notions and results:

- we introduced different confluence properties for topological rewriting systems
- we provided counter-examples for converse strength implications
- we showed that **topological confluence** is equivalent to **infinitary confluence** for formal power series thanks to the topological closure of ideals

Further works:

- prove or disprove the **chains conjecture**
- find a generalised Newman's lemma in the topological setting
- develop computational tools to manipulate Taylor series to ultimately investigate formal solutions of PDEs

THANK YOU FOR LISTENING!