Topological closure of formal powers series ideals and application to topological rewriting theory

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Abstract

We investigate formal power series ideals and their relationship to topological rewriting theory. Since commutative formal power series algebras are Zariski rings, their ideals are closed for the adic topology defined by the maximal ideal generated by the indeterminates. We provide a constructive proof of this result which, given a formal power series in the topological closure of an ideal, consists in computing a cofactor representation of the series with respect to a standard basis of the ideal. We apply this result to topological rewriting theory. In this context, two natural notions of confluence arise: topological confluence and infinitary confluence. We give explicit examples illustrating that in general, infinitary confluence is a strictly stronger notion that topological confluence. Using topological closure of ideals, we finally show that in the context of rewriting theory on commutative formal power series, infinitary and topological confluence are equivalent.

Keywords: Complete rings, formal power series, topological rewriting.

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1. INTRODUCTION

Algebraic rewriting systems are computational models used to prove algebraic properties through rewriting reasoning. For that, one considers a presentation by generators and relations of an algebraic system, *e.g.*, a monoid or an algebra, and associate to each relation a rewriting rule, consisting in simplifying one term of the relation into the other terms of the relation. When the underlying algebraic system is equipped with the discrete topology, two fundamental rewriting properties are termination, *i.e.*, there is no infinite rewriting sequence, and confluence, *i.e.*, whenever two finite rewriting sequences diverge from a common term, then these sequences can be extended by finitely many rewriting steps to reach a common term. When these two properties hold together, rewriting theory provide effective methods for computing linear bases, Hilbert series, homotopy bases or free resolutions [1, 8, 9, 12], and obtain constructive proofs of coherence theorems, from which we deduce an explicit description of the action of a monoid on a category [5], or of homological properties such as finite derivation type, finite homological type [10, 14], or Koszulness [13].

Topological rewriting theory is an extension of discrete rewriting, where the underlying set of terms admits a non-discrete topology. Such topological rewriting systems appear in computer science, in the context of rewriting over infinitary Σ/λ-terms [4, 11], and in abstract algebra, in the context of rewriting over commutative formal power series [3]. With this topological framework, it is natural to consider not only finite rewriting sequences, but also rewriting sequences that converge for the underlying topology, which brings us to two different notions of confluence: topological and infinitary confluence. Each of these notions allows us to extend diverging rewriting sequences by infinite rewriting sequences having a common term as a limit. However, the rewriting sequences diverging from the same term have different natures: they are assumed to be finite for topological confluence, and they are converging to limits when one deals with infinitary confluence. Infinitary confluence is a strictly stronger property in general, explicit examples are given in Section 4.1 of the present paper.

Our objective is to show that in the context of rewriting over commutative formal power series, topological and infinitary confluence are actually equivalent properties.

Topological closure of commutative formal power series ideals. Proving that infinitary confluence and topological confluence are equivalent in the context of commutative formal power series requires to establish that an ideal of a formal power series algebra $\mathbb{K}[[x_1, \cdots, x_n]]$ is closed for the $(x_1, \cdots, x_n)$-adic topology. The latter, denoted by τ_δ, is induced by the metric δ which is defined by

$$\delta(f, g) = \frac{1}{2^{\mathrm{val}(f-g)}},$$

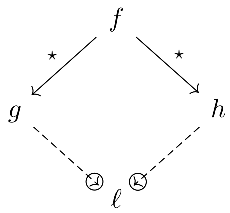
where the valuation $\mathrm{val}(h)$ of h is the smallest degree of a monomial appearing with a non-zero coefficient in h. By a general theorem on Zariski rings [15], all the ideals of $\mathbb{K}[[x_1, \cdots, x_n]]$ are indeed closed for the topology τ_δ. In this paper, we propose a constructive proof of this result, which is based on rewriting theory. Indeed, in order to show that any ideal $I \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ is equal to its topological closure $\overline{I}$, we fix a finite standard basis G of I, *i.e.*, a generating set of I that induces a topologically confluent rewriting system over $\mathbb{K}[[x_1, \cdots, x_n]]$. Such standard bases have the property that their leading monomials, for a given order on monomials, divide leading monomials of all series in I. In Proposition 3.1.1, we show how the definition of the metric δ implies that the set of leading monomials of I and $\overline{I}$ are equal when the order on monomials is compatible with the degree, in the sense that it is increasing with respect to the degree. From this, given a commutative formal power series f in $\overline{I}$, we get a procedure for eliminating monomials in f using G. This procedure constructs at each step a cofactor representation with respect to G that converges to f. More explicitly, for each $s_i \in G$, this yields a coefficient $f_i^{(\infty)} \in \mathbb{K}[[x_1, \cdots, x_n]]$ proving that f is indeed in I, as stated in our first contribution:

Theorem 3.1.6. *Let $I \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ be a formal power series ideal, $G = \{s_1, \cdots, s_\ell\}$ be a finite standard basis of I with respect to a monomial order which is compatible with the degree, and f be an element in the topological closure of I for the $(x_1, \cdots, x_n)$-adic topology. Then, the limit coefficients $\left(f_1^{(\infty)}, \cdots, f_\ell^{(\infty)}\right)$ of f relative to G verify:*

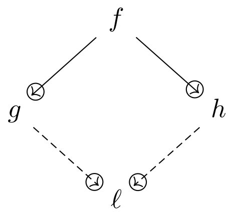
$$f = f_1^{(\infty)} s_1 + \cdots + f_\ell^{(\infty)} s_\ell.$$

In particular, I is closed for the $(x_1, \cdots, x_n)$-adic topology.

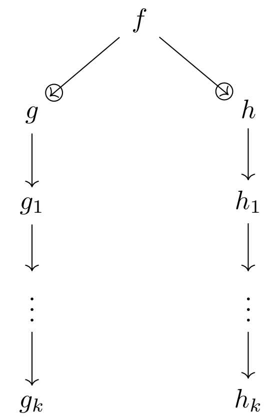
Infinitary and topological confluence for commutative formal power series. As stated above, infinitary confluence implies topological confluence. In order to show the equivalence of the two notions for rewriting systems over commutative formal power series, we thus have to show the converse. For that, we consider a topological rewriting system on $\mathbb{K}[[x_1, \cdots, x_n]]$, where the rewriting relation is induced by a subset $G \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ and an order on monomials that is compatible with the degree. In other words, we have a rewriting step $f \to h$, if we can substitute in f a leading monomial of an element of G and replace it with the corresponding remainder to get h. We assume that this rewriting relation is topologically confluent, meaning that whenever a commutative formal power series f rewrites after finitely many rewriting steps into two commutative formal power series g and h, then the latter rewrite after possibly infinite rewriting steps into a common limit ℓ. Pictorially, we have



where $\overset{\star}{\to}$ and $\to\!\!\!\otimes$ represent finite rewriting sequences and rewriting sequences that have well-defined limits for the $(x_1, \cdots, x_n)$-adic topology τ_δ, respectively. Infinitary confluence is thus represented by



Hence, we have to prove that if we assume topological confluence only, the dashed arrows always exist in the last diagram. The crucial observation is that since commutative formal power series ideals are closed for τ_δ, the elements $f - g$ and $f - h$, and thus also $g - h$, belong to the ideal I generated by G. Since the assumption of topological confluence is equivalent to the fact that G is a standard basis of I, leading monomials of elements of I are always divisible by leading monomials of elements of G, so that we can rewrite simultaneously g and h as long as we obtain different results:



The rewriting process stops if $g_k = h_k$ at some step k, in such case the dashed arrows are constructed. If not, we show that the sequences $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ are in fact Cauchy sequences for the metric δ and thus, have limits. The last part of the proof is that these two limits are equal and our main result is stated as follows.

Theorem 4.2.2. *Let $G \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ be a set of formal power series, $<$ be a monomial order that is compatible with the degree, and $(\mathbb{K}[[x_1, \cdots, x_n]], \tau_\delta, \to)$ be the induced topological rewriting system. Then, $\to$ is τ_δ-confluent if and only if it is infinitary confluent.*

2. CONVENTIONS AND NOTATIONS

In this section, we recall the construction of algebras of commutative formal power series and the notion of standard basis for an ideal of commutative formal power series.

In the next section, we will consider non-commutative formal power series. From now on, we will drop the adjective ”commutative” when we consider commutative formal power series, and simply specify the adjective “non-commutative” when we consider non-commutative formal power series.

2.1. Formal power series algebras

Throughout this paper, we fix a positive integer n, a set of indeterminates $\{x_1, \cdots, x_n\}$, and a field $\mathbb{K}$. We denote by $[x_1, \cdots, x_n]$ the free monoid generated by $\{x_1, \cdots, x_n\}$ and by $\mathbb{K}[x_1, \cdots, x_n]$ the algebra of polynomials with indeterminates $\{x_1, \cdots, x_n\}$ and coefficients in $\mathbb{K}$. We denote by $\deg(m)$ the degree of a monomial $m \in [x_1, \cdots, x_n]$, *i.e.*, if $m = x_1^{\mu_1} \cdots x_n^{\mu_n}$ with non-negative integers $\mu_1, \cdots, \mu_n$, then $\deg(m) = \mu_1 + \cdots + \mu_n$. Given a polynomial $f \in \mathbb{K}[x_1, \cdots, x_n]$, we denote by $\langle f, m\rangle$ the coefficient of m in f. Let $\mathrm{supp}(f)$ be the *support* of f, defined as the set of monomials that occur in f:

$$\mathrm{supp}(f) := \{m \in [x_1, \cdots, x_n],\ \langle f, m\rangle \neq 0\}.$$

We denote by $\mathbb{K}[[x_1, \cdots, x_n]]$ the algebra of formal powers series with indeterminates $\{x_1, \cdots, x_n\}$ and coefficients in $\mathbb{K}$, defined as the Cauchy completion of $\mathbb{K}[x_1, \cdots, x_n]$ for the distance δ defined by

$$\forall f,\ g \in \mathbb{K}[x_1, \cdots, x_n], \quad \delta(f, g) := \frac{1}{2^{\mathrm{val}(f-g)}}, \tag{1}$$

where, for $h \in \mathbb{K}[x_1, \cdots, x_n]$, $\mathrm{val}(h)$ denotes the lowest degree of the monomials that are in $\mathrm{supp}(h)$. The coefficient of $m \in [x_1, \cdots, x_n]$ in a formal power series $f \in \mathbb{K}[[x_1, \cdots, x_n]]$ is still written $\langle f, m\rangle$.

2.2. Standard bases of formal power series ideals

Throughout this section, we fix a *monomial order* $<$ on $[x_1, \cdots, x_n]$, that is a well-order such that for all monomials $m_1, m_2, m_3 \in [x_1, \cdots, x_n]$, we have the implication

$$m_1 < m_2 \Rightarrow m_1 \cdot m_3 < m_2 \cdot m_3.$$

We say that $<$ is *compatible with the degree*, if for all monomials m_1, m_2 such that $\deg(m_1) < \deg(m_2)$ it follows that $m_1 < m_2$. We denote by $<_{\text{op}}$ the opposite order of $<$. For a non-zero $f \in \mathbb{K}[[x_1, \cdots, x_n]]$, we define the *leading monomial* and the *leading coefficient*, written $\text{lm}(f)$ and $\text{lc}(f)$, as the greatest monomial for $<_{\text{op}}$ that occur in f and the coefficient of $\text{lm}(f)$ in f, respectively, *i.e.*, we have:

$$\text{lm}(f) := \max_{<_{\text{op}}} \text{supp}(f), \qquad \text{lc}(f) := \langle f, \text{lm}(f) \rangle.$$

We define the *remainder* of f as:

$$r(f) := \text{lc}(f)\,\text{lm}(f) - f.$$

Notice that either $r(f) = 0$ or $\text{lm}(r(f)) <_{\text{op}} \text{lm}(f)$. Moreover, we verify the following properties:

- $\forall f \in \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\},\ \forall m \in [x_1, \cdots, x_n],\ \text{lm}(m \times f) = m \cdot \text{lm}(f)$,
- $\forall f, g \in \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\},\ \text{lm}(f + g) <_{\text{op}} \max_{<_{\text{op}}}\{\text{lm}(f), \text{lm}(g)\}$,
- $\forall f \in \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\},\ \forall \lambda \in \mathbb{K} \setminus \{0\},\ \text{lm}(\lambda f) = \text{lm}(f)$.

Definition 2.2.1. Let $I \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ be a formal power series ideal and let $<$ be a monomial order on $[x_1, \cdots, x_n]$. A *standard basis* of I with respect to $<$ is a subset $G \subseteq I$ such that for every non-zero formal power series $f \in I$, there exists $g \in G$ such that $\text{lm}(g)$ divides $\text{lm}(f)$.

Recall from [7, Section 6.4] that for every ideal $I \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$, there always exists a finite standard basis of I.

3. TOPOLOGICAL PROPERTIES OF FORMAL POWER SERIES IDEALS

In this section, we provide a constructive proof of the fact that formal power series ideals are closed for the $(x_1, \cdots, x_n)$-adic topology, induced by the metric δ defined in the previous section. This is a particular case of a more general result about Zariski rings [15]. We also recall from [6] that for non-commutative formal power series, it exist ideals that are not closed.

3.1. Closure in the commutative case

Let us consider an ideal $I \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ and the $(x_1, \cdots, x_n)$-adic topology τ_δ, induced by the metric δ. We shall prove that I is closed for τ_δ. To do that, we will work with an arbitrary standard basis of I, so that we fix a monomial order $<$ on $[x_1, \cdots, x_n]$ and a finite standard basis $G = \{s_1, \cdots, s_\ell\}$ of I with respect to $<$.

Given a subset $S \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$, we denote by $\text{lm}(S)$ the set of leading monomials of non-zero elements of S with respect to $<_{\text{op}}$:

$$\text{lm}(S) := \{\text{lm}(f),\ f \in S \setminus \{0\}\}.$$

We start by showing the following result, in which $\overline{I}$ denotes the topological closure of I for τ_δ.

Proposition 3.1.1. *If the monomial order $<$ is compatible with the degree, then* $\mathrm{lm}(I) = \mathrm{lm}(\overline{I})$.

Proof. Since $I \subseteq \overline{I}$, we clearly have $\mathrm{lm}(I) \subseteq \mathrm{lm}(\overline{I})$. Let us show the converse inclusion. Let $f \in \overline{I}$ be a non-zero formal power series. Then there exist formal power series in I arbitrarily close to f for the metric δ, in particular, there exists $g \in I$ such that

$$\delta(g, f) < \frac{1}{2^{\deg(\mathrm{lm}(f))}},$$

meaning that $\deg(\mathrm{lm}(g - f)) > \deg(\mathrm{lm}(f))$. Since the order $<$ is assumed to be compatible with the degree, we get $\mathrm{lm}(g - f) > \mathrm{lm}(f)$. Thus, for any monomial $m \leq \mathrm{lm}(f)$, we have $\langle g - f, m\rangle = 0$, hence, we have $\langle g, m\rangle = \langle f, m\rangle$. By definition of $\mathrm{lm}(f)$, it follows that, on one hand, g is non-zero and, on the other hand, $\mathrm{lm}(g) = \mathrm{lm}(f)$. Hence, $\mathrm{lm}(f) \in \mathrm{lm}(I)$, which ends the proof. □

Remark 3.1.2. In Theorem 3.1.6, we will show that I is closed, from which we get that $\mathrm{lm}(I) = \mathrm{lm}(\overline{I})$ is true for any monomial order. However, we were not able to provide a proof of this fact that works for monomial orders that are not assumed to be compatible with the degree.

In order to show that I is closed in $\mathbb{K}[[x_1, \cdots, x_n]]$, we have to show that $\overline{I}$ is included in I, *i.e.*, every $f \in \overline{I}$ belongs to I. For that, we are going to use Proposition 3.1.1 to construct, for any formal power series f in $\overline{I}$, a tuple $(f_i)_{1\leq i\leq \ell}$ of formal power series such that $f = f_1 s_1 + \cdots + f_\ell s_\ell$. This will suffice to prove that f belongs to I since the s_i are elements of I.

Fix $f \in \overline{I}$ and assume from now on that the monomial order is compatible with the degree. Note that the latter assumption is not restrictive since we can choose the monomial order we want and work with it to achieve our goal. Let us construct inductively:

- a sequence $\mathbb{N} \ni k \mapsto \left(f_i^{(k)}\right)_{1\leq i\leq \ell}$ of tuples of formal power series,
- a sequence $(F_k)_{k\in\mathbb{N}}$ of formal power series in $\overline{I}$ and $(m_k)_{k\in\mathbb{N}}$ the corresponding sequence of leading monomials, *i.e.*, for all $k \in \mathbb{N}$, $m_k := \mathrm{lm}(F_k)$,
- a sequence $(i_k)_{k\in\mathbb{N}}$ of indices in $[\![1, \cdots, \ell]\!]$ and a sequence $(q_k)_{k\in\mathbb{N}}$ of monomials.

Base case: we let, for any $i \in [\![1, \cdots, \ell]\!]$, $f_i^{(0)} := 0$. Notice how we obtain $f - \sum_{i=1}^{\ell} f_i^{(0)} s_i \in \overline{I}$.

Induction step: assume the sequence is defined up to and including $k \in \mathbb{N}$ in such a way that:

$$F_k := f - \sum_{i=1}^{\ell} f_i^{(k)} s_i \in \overline{I}.$$

- If $F_k = 0$, we then have $f = f_1 s_1 + \cdots + f_\ell s_\ell$, where $s_i \in I$ and $f_i = f_i^{(k)} \in \mathbb{K}[[x_1, \cdots, x_n]]$. Hence, $f \in I$ and it is over.
- Otherwise, the leading monomial $m_k := \mathrm{lm}(F_k)$ is well-defined. Since $F_k \in \overline{I}$ and $<$ is compatible with the degree, we have $m_k \in \mathrm{lm}(I)$ by Proposition 3.1.1. Then, as G is a standard basis of I with respect to $<$, there exists $i_k \in [\![1, \cdots, \ell]\!]$ and $q_k \in [x_1, \cdots, x_n]$ such that:

$$m_k = \mathrm{lm}(s_{i_k}) \cdot q_k. \tag{2}$$

We then defined the $(k + 1)$'th tuple in the sequence as:

$$\forall i \in [\![1, \cdots, \ell]\!] \setminus \{i_k\},\ f_i^{(k+1)} := f_i^{(k)},$$

and:

$$f_{i_k}^{(k+1)} := f_{i_k}^{(k)} + \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k.$$

Notice that:

$$\begin{aligned}
F_{k+1} &:= f - \sum_{i=1}^{\ell} f_i^{(k+1)} s_i, \\
&= f - f_{i_k}^{(k)} s_{i_k} - \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k \times s_{i_k} - \sum_{\substack{i=1 \\ i \neq i_k}}^{\ell} f_i^{(k)} s_i, \\
&= \left(f - \sum_{i=1}^{\ell} f_i^{(k)} s_i \right) - \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k \times s_{i_k}, \\
&= F_k - \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k \times s_{i_k}.
\end{aligned}$$

By induction hypothesis, we have $F_k \in \overline{I}$ and since $s_{i_k} \in I \subseteq \overline{I}$, we get $F_{k+1} \in \overline{I}$.

If at any step k, we get $F_k = 0$, we obtain $f \in I$ as explained above. Thus, assume from now on that, for all $k \in \mathbb{N}$, we have $F_k \neq 0$.

Lemma 3.1.3. *The sequence of monomials $(m_k)_{k \in \mathbb{N}}$ is strictly decreasing for the opposite order $<_{\mathrm{op}}$.*

Proof. By construction of i_k and q_k, we find:

$$q_k \times s_{i_k} = \mathrm{lc}(s_{i_k}) \left(\mathrm{lm}(s_{i_k}) \cdot q_k \right) - q_k \times r(s_{i_k}) = \mathrm{lc}(s_{i_k}) m_k - q_k \times r(s_{i_k}).$$

Hence, since $F_k = \mathrm{lc}(F_k) m_k - r(F_k)$:

$$\begin{aligned}
F_{k+1} &= F_k - \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k \times s_{i_k}, \\
&= \mathrm{lc}(F_k) m_k - r(F_k) - \mathrm{lc}(F_k) m_k + \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k \times r(s_{i_k}), \\
&= \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k \times r(s_{i_k}) - r(F_k).
\end{aligned}$$

And thus, from the properties of leading monomials given earlier, we get

$$m_{k+1} = \mathrm{lm}(F_{k+1}) = \mathrm{lm}\left(\frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k \times r(s_{i_k}) - r(F_k) \right) \leq_{\mathrm{op}} \max_{<_{\mathrm{op}}} \left\{ q_k \cdot \mathrm{lm}(r(s_{i_k})), \mathrm{lm}(r(F_k)) \right\},$$

unless, either $r(s_{i_k})$ or $r(F_k)$ is zero, in which case one of the leading monomial is ill-defined. In that case, we have $m_{k+1} \leq_{\mathrm{op}} r$ where r is either $q_k \cdot \mathrm{lm}(r(s_{i_k}))$ or $\mathrm{lm}(r(F_k))$ according to which of the remainders is well-defined; note how both remainders cannot be simultaneously zero since $F_{k+1} \neq 0$. It then follows from the properties of remainders that:

- $q_k \cdot \mathrm{lm}(r(s_{i_k})) <_{\mathrm{op}} q_k \cdot \mathrm{lm}(s_{i_k}) = m_k$ if $r(s_{i_k})$ is non-zero and,
- $\mathrm{lm}(r(F_k)) <_{\mathrm{op}} \mathrm{lm}(F_k) = m_k$ if $r(F_k)$ is non-zero.

Hence, in the end, we get: $m_{k+1} <_{\mathrm{op}} m_k$. □

Let us now define a family of ℓ sequences $\left(q_1^{(k)}\right)_{k\in\mathbb{N}}, \cdots, \left(q_\ell^{(k)}\right)_{k\in\mathbb{N}}$ containing monomials or zeros as follows: for any $i \in [\![1, \cdots, \ell]\!]$ define $q_i^{(0)} = 0$ and then for any $k \in \mathbb{N}$, fix:

$$q_i^{(k+1)} := \frac{\mathrm{lc}(s_{i_k})}{\mathrm{lc}(F_k)} \left(f_i^{(k+1)} - f_i^{(k)}\right).$$

Hence, for any $k \in \mathbb{N}$ and $i \in [\![1, \cdots, \ell]\!]$, two options arise:

- either $i \neq i_k$, in which case $f_i^{(k+1)} = f_i^{(k)}$ and thus $q_i^{(k+1)} = 0$,
- or $i = i_k$ and then $f_i^{(k+1)} = f_i^{(k)} + \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_k$, and so $q_i^{(k+1)} = q_k$.

In other words, for any $i \in [\![1, \cdots, \ell]\!]$ and any $k \in \mathbb{N}$, the monomial $q_i^{(k+1)}$ is non-zero if and only if the monomial m_k has been "eliminated" at Step (2) by choosing $\mathrm{lm}(s_i)$. These sequences then verify for any $i \in [\![1, \cdots, \ell]\!]$ and any $k \in \mathbb{N}$:

$$f_i^{(k)} = \sum_{\substack{j=1 \\ i_j = i}}^{k} \langle f_i^{(k)}, q_i^{(j)} \rangle q_i^{(j)}.$$

This exhibits, with i and k fixed, that the subfamily $Q_i^{(k)}$ of non-zero elements from $\left(q_i^{(j)}\right)_{1\le j\le k}$ is exactly the support of the formal power series $f_i^{(k)}$. Moreover, this family $Q_i^{(k)}$ is strictly decreasing for the opposite order $<_{\mathrm{op}}$. Indeed, either the family $Q_i^{(k)}$ is empty, either it contains a single monomial, in both cases it is then obvious, or $Q_i^{(k)} = \left(q_i^{(j_1)}, \cdots, q_i^{(j_r)}\right)$ contains $r \geq 2$ monomials where $j_1 < \cdots < j_r$ are indices in $[\![1, \cdots, k]\!]$. Let us show that for any $s \in [\![1, \cdots, r-1]\!]$, we have $q_i^{(j_s)} >_{\mathrm{op}} q_i^{(j_{s+1})}$. Indeed, by definition, we have $q_i^{(j_s)} = q_{j_s - 1}$ and $q_i^{(j_{s+1})} = q_{j_{s+1}-1}$. But, by construction of the family $Q_i^{(k)}$, the indices chosen at the step (2) for $k_s := j_s - 1$ and $k'_s := j_{s+1} - 1$ verify $i_{k_s} = i_{k'_s} = i$, and so we have the equalities $m_{k_s} = \mathrm{lm}(s_i) \cdot q_{k_s}$ and $m_{k'_s} = \mathrm{lm}(s_i) \cdot q_{k'_s}$. But $k_s < k'_s$ and the sequence $(m_k)_{k\in\mathbb{N}}$ is strictly decreasing for $<_{\mathrm{op}}$ from Lemma 3.1.3, and so $\mathrm{lm}(s_i) \cdot q_{k_s} >_{\mathrm{op}} \mathrm{lm}(s_i) \cdot q_{k'_s}$, which implies $q_{k_s} >_{\mathrm{op}} q_{k'_s}$.

For $i \in [\![1, \cdots, \ell]\!]$ fixed, $\left(Q_i^{(k)}\right)_{k\in\mathbb{N}}$ satisfies for all $k_2 > k_1$, either $Q_i^{(k_1)} = Q_i^{(k_2)}$ or $Q_i^{(k_1)} \subsetneq Q_i^{(k_2)}$ in such a way that any element in the second family which is not in the first one is smaller for $<_{\mathrm{op}}$ than any element of the family $Q_i^{(k_1)}$. In other words, we can construct the family $Q_i^{(\infty)} = \bigcup_{k\geq 1} Q_i^{(k)}$ which is strictly decreasing for $<_{\mathrm{op}}$.

These facts on these new sequences entail the following proposition.

Proposition 3.1.4. *For any $i \in [\![1, \cdots, \ell]\!]$ fixed, the sequence $\left(f_i^{(k)}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence.*

Proof. Using the definition of the sequence $\left(q_i^{(k)}\right)_{k\in\mathbb{N}}$, we obtain

$$f_i^{(k+1)} - f_i^{(k)} = \frac{\mathrm{lc}(F_k)}{\mathrm{lc}(s_{i_k})} q_i^{(k+1)},$$

in such a way that for any positive integers $k_1 < k_2$, we get:

$$f_i^{(k_2)} - f_i^{(k_1)} = \sum_{j=k_1}^{k_2-1} \left(f_i^{(j+1)} - f_i^{(j)}\right) = \sum_{j=k_1}^{k_2-1} \frac{\mathrm{lc}(F_j)}{\mathrm{lc}(s_{i_j})} q_i^{(j+1)}. \tag{3}$$

Thus, either the sequence $\left(f_i^{(k)}\right)_{k\in\mathbb{N}}$ is stationary, it is the case if and only if the index i has been chosen only finitely many times among the infinite times we went through the step (2), and in which case the sequence is obviously a Cauchy sequence. Or the sequence is not stationary and then there will always exist $k_2 > k_1$ such that $\delta\left(f_i^{(k_2)}, f_i^{(k_1)}\right) > 0$. However, for any real number $\varepsilon > 0$, we can fix

$$K_\varepsilon := \min\left\{k \in \mathbb{N} \setminus \{0\},\ q_i^{(k)} \neq 0 \text{ and } \deg\left(q_i^{(k)}\right) > \log_2\left(\frac{1}{\varepsilon}\right)\right\}.$$

The integer K_ε is well-defined since $\left(f_i^{(k)}\right)_{k\in\mathbb{N}}$ is assumed non-stationary and the family $Q_i^{(\infty)}$, which contains all non-zero $q_i^{(k)}$, is strictly decreasing for $<_{\text{op}}$, so that the degrees of the monomials from $Q_i^{(\infty)}$ are unbounded because there are finitely many variables and the order is compatible with the degree. It follows that, for any $k_2 > k_1 \geq K_\varepsilon$, either $f_i^{(k_2)} = f_i^{(k_1)}$ in which case $\delta\left(f_i^{(k_2)}, f_i^{(k_1)}\right) = 0 < \varepsilon$, or, the monomial $\operatorname{lm}\left(f_i^{(k_2)} - f_i^{(k_1)}\right)$ is well-defined and satisfies, by formula (3):

$$\operatorname{lm}\left(f_i^{(k_2)} - f_i^{(k_1)}\right) \leq \max_{<_{\text{op}}}\left\{q_i^{(j+1)},\ q_i^{(j+1)} \neq 0 \text{ and } k_1 \leq j < k_2\right\}.$$

This maximum is well-defined because the set of the right hand side of the inequality is finite and non-empty since $f_i^{(k_2)} \neq f_i^{(k_1)}$. Denoting by $q_i^{(j_0+1)}$ that maximum, we get $j_0 + 1 > k_1 \geq K_\varepsilon$. Since the family $Q_i^{(\infty)}$ is strictly decreasing for $<_{\text{op}}$, we have $q_i^{(j_0+1)} <_{\text{op}} q_i^{(K_\varepsilon)}$, and thus

$$\deg\left(q_i^{(j_0+1)}\right) \geq \deg\left(q_i^{(K_\varepsilon)}\right) > \log_2\left(\frac{1}{\varepsilon}\right),$$

because the order is compatible with the degree. Since $q_i^{(j_0+1)} = \operatorname{lm}\left(f_i^{(k_2)} - f_i^{(k_1)}\right)$, the formula means

$$\delta\left(f_i^{(k_2)}, f_i^{(k_1)}\right) < \varepsilon.$$

Since this is true for every $k_2 > k_1$ and every $\varepsilon > 0$, the sequence $\left(f_i^{(k)}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence. ☐

From Proposition 3.1.4, since $\mathbb{K}[[x_1, \cdots, x_n]]$ is Cauchy-complete, for any $i \in [\![1, \cdots, \ell]\!]$, the sequence $\left(f_i^{(k)}\right)_{k\in\mathbb{N}}$ converges to a limit we denote $f_i^{(\infty)}$.

Definition 3.1.5. The elements $\left(f_1^{(\infty)}, \cdots, f_\ell^{(\infty)}\right)$ are called the *limit coefficients of f relative to G.*

We are now able to prove the main result of the section.

Theorem 3.1.6. *Let $I \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ be a formal power series ideal, $G = \{s_1, \cdots, s_\ell\}$ be a finite standard basis of I with respect to a monomial order which is compatible with the degree, and f be an element in the topological closure of I for the $(x_1, \cdots, x_n)$-adic topology. Then, the limit coefficients $\left(f_1^{(\infty)}, \cdots, f_\ell^{(\infty)}\right)$ of f relative to G verify:*

$$f = f_1^{(\infty)} s_1 + \cdots + f_\ell^{(\infty)} s_\ell.$$

In particular, I is closed for the $(x_1, \cdots, x_n)$-adic topology.

Proof. By continuity of algebraic operations, the sequence $\left(f - \sum_{i=1}^{\ell} f_i^{(k)} s_i\right)_{k\in\mathbb{N}}$ converges to

$$\lim_{k\to\infty}\left(f - \sum_{i=1}^{\ell} f_i^{(k)} s_i\right) = f - \sum_{i=1}^{\ell} f_i^{(\infty)} s_i.$$

But then, on one hand:

$$\lim_{k\to\infty}\delta\left(f-\sum_{i=1}^{\ell}f_i^{(k)}s_i,0\right)=\frac{1}{2^{\lim_{k\to\infty}\deg\left(\operatorname{lm}\left(f-\sum_{i=1}^{\ell}f_i^{(k)}s_i\right)\right)}}=\frac{1}{2^{\lim_{k\to\infty}\deg(m_k)}}.$$

On the other hand, Lemma 3.1.3 shows that the sequence $(m_k)_{k\in\mathbb{N}}$ is strictly decreasing for $<_{\mathrm{op}}$. Thus, since the order is compatible with the degree, the sequence $(\deg(m_k))_{k\in\mathbb{N}}$ tends to infinity, and thus, we have:

$$0=\lim_{k\to\infty}\delta\left(f-\sum_{i=1}^{\ell}f_i^{(k)}s_i,0\right)=\delta\left(\lim_{k\to\infty}\left(f-\sum_{i=1}^{\ell}f_i^{(k)}s_i\right),0\right)=\delta\left(f-\sum_{i=1}^{\ell}f_i^{(\infty)}s_i,0\right).$$

We finally get $f=f_1^{(\infty)}s_1+\cdots+f_\ell^{(\infty)}s_\ell$, which proves that f belongs to I. □

3.2. A counter-example in the non-commutative case

In this section, we recall from [6] that ideals of non-commutative formal power series algebras are not necessarily closed. We also explain why the proof given in the previous section does not translate in the non-commutative case.

Non-commutative formal power series are constructed in the same way as commutative ones, where we replace the polynomial algebra $\mathbb{K}[x_1,\cdots,x_n]$ by the tensor algebra $\mathbb{K}\langle x_1,\cdots,x_n\rangle$ over the vector space with basis $\{x_1,\cdots,x_n\}$. As in the commutative case, the distance δ on $\mathbb{K}\langle x_1,\cdots,x_n\rangle$ is defined by formula $\delta(f,g):=2^{-\operatorname{val}(f-g)}$, and $\mathbb{K}\langle\langle x_1,\cdots,x_n\rangle\rangle$ is the Cauchy-completion of $\mathbb{K}\langle x_1,\cdots,x_n\rangle$ for δ.

Let $\mathbb{K}\langle\langle x,y\rangle\rangle$ be the algebra of non-commutative formal power series in two variables x and y and consider the two-sided ideal I generated by y. Then, from [6, Lemma 1.2], the series

$$\sum_{n\in\mathbb{N}}x^nyx^n,$$

does not belong to I, but it belongs to the topological closure $\overline{I}$ of I for the (x,y)-adic topology induced by δ. That shows that I is not closed in $\mathbb{K}\langle\langle x,y\rangle\rangle$ for the (x,y)-adic topology.

Notice how this situation would not arise in the commutative case. Indeed, we have for commutative monomials:

$$\sum_{n\in\mathbb{N}}x^nyx^n=\sum_{n\in\mathbb{N}}x^{2n}y=\left(\sum_{n\in\mathbb{N}}x^{2n}\right)y\in I.$$

The first point where the procedure described in the previous section fails to translate in the non-commutative case is that a non-commutative formal power series ideal does not necessarily admit a finite standard basis. However, even if such a finite standard basis exists, our procedure still does not translate in the non-commutative setting: indeed, Step (2) would require two monomials q_k^{left} and q_k^{right} to factorise the leading monomial m_k instead of a single one; similarly, we would need two formal power series $f_i^{\mathrm{left}(k)}$ and $f_i^{\mathrm{right}(k)}$ to obtain the relation at each step k:

$$f-\sum_{i=1}^{\ell}f_i^{\mathrm{left}(k)}s_if_i^{\mathrm{right}(k)}\in\overline{I}.$$

The process would then yield two limit formal power series that describe f in terms of the s_i. However, it would not be possible to factorise *a priori* the s_i in such a way that f becomes a combination of them with formal power series coefficients, *i.e.*, f would not be in the ideal I.

4. APPLICATION TO TOPOLOGICAL REWRITING THEORY

In this section, we show that topological confluence and infinitary confluence are equivalent notions for rewriting over formal power series. These two properties provide characterisations of standard bases.

4.1. Topological and infinitary confluence

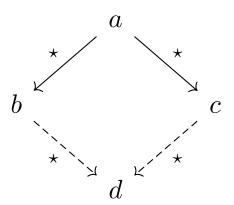
In this section, we recall the definition of topological rewriting systems as well as various notions of confluence associated with them. We also show how these notions are related to each other.

We recall that a *topological rewriting system* is a triple $(A, \tau, \to)$, where (A, τ) is a topological space and $\to \subseteq A \times A$ is a binary relation on A, called *rewriting relation.* We denote by $\overset{\star}{\to}$ the symmetric transitive closure of $\to$ and $-\!\!\circledast$ the topological closure of $\overset{\star}{\to}$ for the product topology $\tau^A_{\text{dis}} \times \tau$ on $A \times A$, where τ^A_{dis} is the discrete topology on A. In other words, we have $a \overset{\star}{\to} b$ if and only if there exists an integer $k \in \mathbb{N}$ and elements $a_0, \cdots, a_k \in A$, such that $a = a_0 \to a_1 \to \cdots \to a_k = b$. The integer k is called the *length* of the sequence and the case $k = 0$ means that $a = b$. Moreover, we have $a -\!\!\circledast\, b$ if and only if every neighbourhood V of b for τ contains $b' \in V$ such that $a \overset{\star}{\to} b'$. An element $a \in A$ is called a *normal form* for $\to$ if whenever we have $a -\!\!\circledast\, b$, then $b = a$.

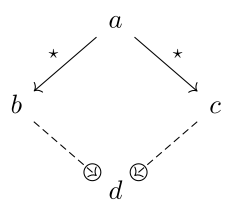
Remark 4.1.1. In the case of discrete rewriting, *i.e.*, when $\tau = \tau^A_{\text{dis}}$, a normal form for $\to$ is a $a \in A$ such that there is no $b \in A$ such that $a \to b$. When there is no $a \in A$ such that $a \to a$, then our definition of normal form is stronger than the one used in the discrete case, since $a \to b$ implies $a -\!\!\circledast\, b$. In general, the discrete definition of normal form is strictly weaker than ours. Indeed, if $\tau = \{\emptyset, A\}$ and the rewriting relation $\to$ is empty, then every $a \in A$ is a normal form in the discrete sense but not for ours, since $a -\!\!\circledast\, b$ for every $b \in A$. However, in the case where (A, τ) is a Hausdorff space and there is no $a \in A$ such that $a \to a$, then the two definitions are equivalent. Indeed, let $a \in A$ be a normal form in the discrete sense and let $b \in A$ such that $a -\!\!\circledast\, b$. Then, every neighbourhood of b contains a b' such that $a \overset{\star}{\to} b'$. But since a is a normal form in the discrete sense, we have $a = b'$, so that a belongs to all the neighbourhoods of b. Since A is Hausdorff, we get that $a = b$ and so a is also a normal form for our definition.

Definition 4.1.2. Let $(A, \tau, \to)$ be a topological rewriting system.

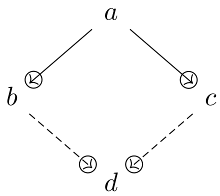
1. The rewriting relation $\to$ is said to be *confluent* if for every $a, b, c \in A$ such that we have $a \overset{\star}{\to} b$ and $a \overset{\star}{\to} c$, then there exists $d \in A$ such that $b \overset{\star}{\to} d$ and $c \overset{\star}{\to} d$:



2. The rewriting relation $\to$ is said to be τ-*confluent* if for every $a, b, c \in A$ such that we have $a \overset{\star}{\to} b$ and $a \overset{\star}{\to} c$, then there exists $d \in A$ such that $b -\!\!\circledast\, d$ and $c -\!\!\circledast\, d$:



3. The rewriting relation $\to$ is said to be *infinitary confluent* if for every $a, b, c \in A$ such that we have $a \xrightarrow{\tau} b$ and $a \xrightarrow{\tau} c$, then there exists $d \in A$ such that $b \xrightarrow{\tau} d$ and $c \xrightarrow{\tau} d$:



Since $\overset{\star}{\to} \subseteq \xrightarrow{\tau}$, confluence and infinitary confluence both imply τ-confluence. The converse implications are both false in general. A counter-example of a τ-confluent rewriting relation that is not confluent is given in terms of formal power series in [3, Example 4.1.4]. Examples of τ-confluent rewriting relations that are not infinitary confluent are given thereafter.

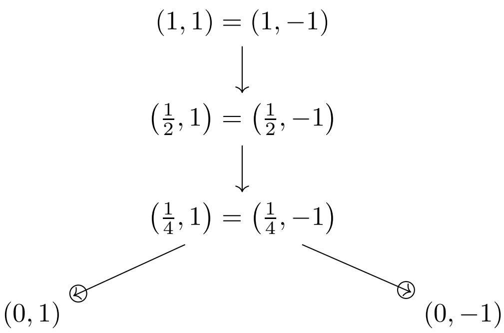
Example 4.1.3. First, consider the real line with two origins, defined as the set

$$X := \Big(\mathbb{R} \times \{\pm 1\}\Big) / \sim,$$

where $\sim$ is the equivalence relation generated by $(x, 1) \sim (x, -1)$, whenever $x \neq 0$, and equip X by the quotient topology τ of the induced topology of $\mathbb{R}^2$ over $\mathbb{R} \times \{\pm 1\}$. Consider the rewriting relation $\to$ defined by

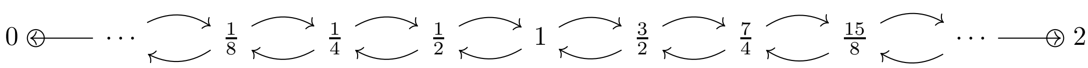
$$\left(\frac{1}{2^n}, 1\right) = \left(\frac{1}{2^n}, -1\right) \to \left(\frac{1}{2^{n+1}}, 1\right) = \left(\frac{1}{2^{n+1}}, -1\right),$$

for every $n \in \mathbb{N}$. This rewriting relation is confluent, hence τ-confluent, because every $(x, y) \in X$ rewrites in one step into at most one element. Moreover, it is not infinitary confluent since we have



and $(0, \pm 1)$ are distinct normal forms for $\to$.
Another counter-example is given by $X = [0, 2] \subseteq \mathbb{R}$ with the usual topology τ and $\to$ is defined by

for every $n \in \mathbb{N}$. Hence, we have:



This rewriting relation is confluent, hence τ-confluent, because every finite rewriting sequence can be reversed, but it is not infinitary confluent since 0 and 2 are distinct normal forms for $\to$.

The main reason why the first rewriting relation given in Example 4.1.3 above is not infinitary confluent is because (X,τ) is not a Hausdorff topological space. For the second counter-example provided in Example 4.1.3, $\to$ is τ-confluent in part because it is cyclic, *i.e.*, we have rewriting loops $a \overset{\star}{\to} a$ of length at least 1. An example of a rewriting relation with no rewriting loop of length at least 1 over a Hausdorff topological space, that is topologically confluent but not infinitary confluent is given in the following.

Example 4.1.4. Consider $X = (\mathbb{N}\cup\{\infty\}) \times (\mathbb{N}\cup\{\infty\})$, equipped with the product topology τ of the order topology over $\mathbb{N}\cup\{\infty\}$. A basis for the order topology over $\mathbb{N}\cup\{\infty\}$ is given by the sets

$$\{n\in\mathbb{N},\ a<n<b\},\qquad \{n\in\mathbb{N},\ n<b\},\qquad \{n\in\mathbb{N}\cup\{\infty\},\ a<n\},$$

where $a\in\mathbb{N}$ and $b\in\mathbb{N}\cup\{\infty\}$. Then, consider the rewriting relation $\to$ on X given by

$$(n,m)\to(n+1,m)\qquad\text{and}\qquad(n,m)\to(n,m+1),$$

whenever $n,m\in\mathbb{N}$. The rewriting relation $\to$ is confluent, hence τ-confluent, because we have a rewriting path $(n,m)\overset{\star}{\to}(n',m')$ if and only if $n,n',m,m'\in\mathbb{N}$ are such that $n\le n'$ and $m\le m'$, so that we have confluence diagrams:

$$\begin{array}{ccc} & (n,m) & \\ \overset{\star}{\swarrow} & & \overset{\star}{\searrow} \\ (n_1,m_1) & & (n_2,m_2) \\ \overset{\star}{\searrow} & & \overset{\star}{\swarrow} \\ & (\max(n_1,n_2),\max(m_1,m_2)) & \end{array}$$

But $\to$ is not infinitary confluent since we have

$$(\infty,0) \overset{\infty}{\longleftarrow} \cdots \longleftarrow (2,0) \longleftarrow (1,0) \longleftarrow (0,0) \longrightarrow (0,1) \longrightarrow (0,2) \longrightarrow \cdots \overset{\infty}{\longrightarrow} (0,\infty)$$

and $(\infty,0)$ and $(0,\infty)$ are distinct normal forms for $\to$.

4.2. Confluence for rewriting on formal powers series

In this section, we recall the construction of topological rewriting systems over formal power series and show that in this setting, τ-confluence and infinitary confluence are equivalent properties.

As in the beginning of the paper, we fix a finite set of indeterminates $\{x_1,\cdots,x_n\}$. For a given monomial order $<$ on $[x_1,\cdots,x_n]$ that is compatible with the degree, and a fixed set G of non-zero formal power series, we define the following rewriting relation $\to$ on $\mathbb{K}[[x_1,\cdots,x_n]]$:

$$\lambda\big(m\cdot\mathrm{lm}(s)\big)+S \to \frac{\lambda}{\mathrm{lc}(s)}\big(m\times r(s)\big)+S,$$

where:

- $\lambda\in\mathbb{K}\setminus\{0\}$ is a non-zero scalar,
- $m\in[x_1,\cdots,x_n]$ is a monomial,
- $s\in G$ is a non-zero formal power series,
- $S\in\mathbb{K}[[x_1,\cdots,x_n]]$ is a formal power series such that $m\cdot\mathrm{lm}(s)\notin\mathrm{supp}(S)$.

We get a topological rewriting system $(\mathbb{K}[[x_1, \cdots, x_n]], \tau_\delta, \to)$, where τ_δ is the $(x_1, \cdots, x_n)$-adic topology induced by the metric δ defined in (1). Recall from [3, Theorem 4.1.3] that a subset G of $\mathbb{K}[[x_1, \cdots, x_n]]$ is a standard basis of the ideal it generates with respect to the monomial order $<$ if and only if the rewriting relation $\to$ is τ_δ-confluent. In the following, we shall show that it is also equivalent to $\to$ being infinitary confluent. We denote by I the formal power series ideal generated by G:

$$I := I(G) \subseteq \mathbb{K}[[x_1, \cdots, x_n]].$$

We first establish the following result, which is the topological adaptation of a well-known result in the context of Gröbner bases theory [2, Theorem 8.2.7]. In the proof of this result, we use the topological closure of ideals of formal power series ideals.

Proposition 4.2.1. *For all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f \xrightarrow{\infty} g$, then $f - g \in I$.*

Proof. First, if $g = f$, then there is nothing to prove. Second, if $f \to g$, then we have

$$f = \lambda(m \cdot \mathrm{lm}(s)) + S, \qquad g = \frac{\lambda}{\mathrm{lc}(s)}(m \times r(s)) + S,$$

for $\lambda \in \mathbb{K} \setminus \{0\}$, $m \in [x_1, \cdots, x_n]$, $s \in G$, and $S \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $m \cdot \mathrm{lm}(s) \notin \mathrm{supp}(S)$. Cancellations ensue in the computation of $f - g$, and we obtain:

$$f - g = \frac{\lambda}{\mathrm{lc}(s)} \times s.$$

But $s \in G \subseteq I$ and I is an ideal, therefore $f - g \in I$. Third, if $f \overset{\star}{\to} g$ and $f \neq g$, then by induction on the length $k \geq 1$ of the rewriting sequence $f = f_0 \to f_1 \to \cdots \to f_k = g$, we have $f - g \in I$. Finally, if we have $f \xrightarrow{\infty} g$, then for every integer $k \in \mathbb{N}$, there exists $f_k \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that

$$\delta(f_k, g) < \frac{1}{2^k},$$

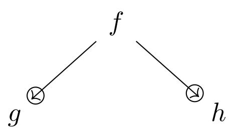
and $f \overset{\star}{\to} f_k$. The sequence $(f_k)_{k\in\mathbb{N}}$ then converges to g. From the third case treated in the proof, for every $k \in \mathbb{N}$, we have $f - f_k \in I$, so that $f - g = \lim_{k\to\infty}(f - f_k)$ belongs to $\overline{I}$. Now, from Theorem 3.1.6, the ideal I is closed, so that $f - g \in I$, which completes the proof. □

We are now in position to prove the main result of this section.

Theorem 4.2.2. *Let $G \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ be a set of formal power series, $<$ be a monomial order that is compatible with the degree, and $(\mathbb{K}[[x_1, \cdots, x_n]], \tau_\delta, \to)$ be the induced topological rewriting system. Then, $\to$ is τ_δ-confluent if and only if it is infinitary confluent.*

Proof. We only have to show that if $\to$ is τ_δ-confluent, then it is also infinitary confluent. Hence, we assume that $\to$ is τ_δ-confluent, which, from [3, Theorem 4.1.3], means that G is a standard basis of the ideal $I = I(G)$ it generates.

Let $f, g, h \in \mathbb{K}[[x_1, \cdots, x_n]]$ be formal power series such that:



In the following, we shall define two sequences $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ of elements of $\mathbb{K}[[x_1, \cdots, x_n]]$ such that $g \overset{\star}{\to} g_k$ and $h \overset{\star}{\to} h_k$, for every $k \in \mathbb{N}$. Then, we will show that these two sequences are Cauchy sequences, hence have limits, and that these two limits are equal, which will conclude the proof. We construct the two sequences by the following recursive procedure.

Base step: we let $g_0 := g$ and $h_0 := h$.

Recursions step: let $k \in \mathbb{N}$ and assume that we have defined rewriting sequences

$$g = g_0 \overset{=}{\to} g_1 \overset{=}{\to} \cdots \overset{=}{\to} g_k, \qquad h = h_0 \overset{=}{\to} h_1 \overset{=}{\to} \cdots \overset{=}{\to} h_k,$$

where $\overset{=}{\to}$ is the reflexive closure of $\to$, *i.e.*, $a \overset{=}{\to} b$ if $a = b$ or if $a \to b$. If we have $g_k - h_k = 0$, then all other terms of the two sequences are equal to $g_k = h_k$, *i.e.*, we define the higher terms of the sequences by $g_l := g_k = h_k$ and $h_l := h_k = g_k$, for every $l \geq k$. Otherwise, if $g_k - h_k \neq 0$, we let

$$m_k := \mathrm{lm}\left(g_k - h_k\right).$$

From Proposition 4.2.1, the formal power series $f - g$, $f - h$, $g - g_k$, and $h - h_k$ belong to I, so that:

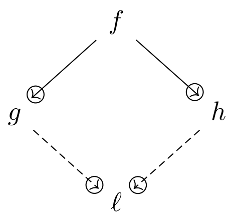
$$g_k - h_k = (g_k - g) + (g - f) + (f - h) + (h - h_k) \in I.$$

Since G is a standard basis of I, there exist $s \in G$ and $m \in [x_1, \cdots, x_n]$ such that $m_k = m \cdot \mathrm{lm}(s)$. Then, we define g_{k+1} and h_{k+1} by rewriting the monomial m_k if possible, *i.e.*, we let

$$\begin{aligned} g_{k+1} &:= g_k - \langle g_k, m_k \rangle \left(m_k - \frac{1}{\mathrm{lc}(s)} (m \times r(s)) \right), \\ h_{k+1} &:= h_k - \langle h_k, m_k \rangle \left(m_k - \frac{1}{\mathrm{lc}(s)} (m \times r(s)) \right). \end{aligned}$$

Indeed, we have either $g_{k+1} = g_k$ or $g_k \to g_{k+1}$, depending on $\langle g_k, m_k \rangle$ is equal to zero or not, hence we have $g_k \overset{=}{\to} g_{k+1}$. In the same manner, we have $h_k \overset{=}{\to} h_{k+1}$. Since $m_k \in \mathrm{supp}(g_k - h_k)$, we cannot have $g_{k+1} = g_k$ and $h_{k+1} = h_k$. Moreover, notice that $m_{k+1} <_{\mathrm{op}} m_k$ because $\langle g_k, m \rangle = \langle h_k, m \rangle$ for every $m_k <_{\mathrm{op}} m$ and m_k, which does not belong to $\mathrm{supp}(g_{k+1}) \cup \mathrm{supp}(h_{k+1})$, rewrites into a series containing monomials that are strictly smaller than m_k for $<_{\mathrm{op}}$, so that $\langle g_{k+1}, m \rangle = \langle h_{k+1}, m \rangle$ for every $m_k \leq_{\mathrm{op}} m$. Hence, we have $m_{k+1} <_{\mathrm{op}} m_k$. Finally, since g_{k+1} and h_{k+1} are equal or are obtained from g_k and h_k by rewriting m_k into a series containing monomials that are strictly smaller than m_k for $<_{\mathrm{op}}$, we have $\langle g_{k+1}, m \rangle = \langle g_k, m \rangle$ and $\langle h_{k+1}, m \rangle = \langle h_k, m \rangle$ for $m_k <_{\mathrm{op}} m$.

If for some $k \in \mathbb{N}$ we have $g_k - h_k = 0$, then letting $\ell = g_k = h_k$, we have $g \overset{\star}{\to} \ell$ and $h \overset{\star}{\to} \ell$, and thus:



Now, assume that for every $k \in \mathbb{N}$, we have $g_k - h_k \neq 0$. Since for each $k \in \mathbb{N}$, we have either $g_{k+1} \neq g_k$ or $h_{k+1} \neq h_k$, then at least one of the two sequences is not stationary. However, both of these sequences are Cauchy sequences. This is obvious if a sequence is stationary. If not, say $(g_k)_{k \in \mathbb{N}}$ is not stationary, we first prove by induction on $i \in \mathbb{N}$ that for every $k \in \mathbb{N}$ and for every monomial m such that $m_k <_{\mathrm{op}} m$:

$$\forall i \in \mathbb{N} : \quad \langle g_k, m \rangle = \langle g_{k+i}, m \rangle. \tag{4}$$

For $i = 0$, this is obvious. Assume (4) for $i \in \mathbb{N}$. Since $\langle g_{k+i+1}, m \rangle = \langle g_{k+i}, m \rangle$ for every monomial m such that $m_{k+i} <_{\mathrm{op}} m$ and since $m_{k+i} \leq_{\mathrm{op}} m_k$, because $(m_k)_{k \in \mathbb{N}}$ is strictly decreasing for $<_{\mathrm{op}}$, we have

$$\langle g_{k+i+1}, m \rangle = \langle g_{k+i}, m \rangle = \langle g_k, m \rangle,$$

for $m_k <_{\mathrm{op}} m$, and the induction is over. As $<_{\mathrm{op}}$ is compatible with the degree, from (4), we have:

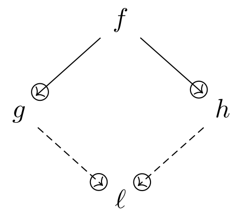
$$\forall i, k \in \mathbb{N}: \quad \delta\left(g_k, g_{k+i}\right) \leq \frac{1}{2^{\deg(m_k)}}.$$

Since $(m_k)_{k\in\mathbb{N}}$ is strictly decreasing, we get that this distance goes to 0 as $k, i \to \infty$, *i.e.*, $(g_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. By the same reasoning, we show that $(h_k)_{k\in\mathbb{N}}$ is also a Cauchy sequence. Using the Cauchy-completeness of $\mathbb{K}[[x_1, \cdots, x_n]]$, the two sequences have limits, denoted by g_∞ and h_∞. Moreover, by construction, we have $g \overset{\circledast}{\to} g_\infty$ and $h \overset{\circledast}{\to} h_\infty$.

It remains to show that $g_\infty = h_\infty$. For every $k \in \mathbb{N}$ and every monomial m such that $m_k <_{\mathrm{op}} m$, we have $\langle g_k, m\rangle = \langle g_\infty, m\rangle$ and $\langle h_k, m\rangle = \langle h_\infty, m\rangle$. Indeed, if for instance $\langle g_k, m\rangle \neq \langle g_\infty, m\rangle$ for some monomial $m_k <_{\mathrm{op}} m$, then from (4), we have $\langle g_{k+i}, m\rangle \neq \langle g_\infty, m\rangle$ for every $i \in \mathbb{N}$, so that

$$\delta\left(g_{k+i}, g_\infty\right) \geq \frac{1}{2^{\deg(m)}},$$

which contradicts the fact that $(g_k)_{k\in\mathbb{N}}$ converges to g_∞. Finally, since the sequence $(m_k)_{k\in\mathbb{N}}$ is strictly decreasing for $<_{\mathrm{op}}$, for an arbitrary fixed monomial m, there exists an index k such that $m_k <_{\mathrm{op}} m$. From the beginning of the paragraph we have $\langle g_k, m\rangle = \langle g_\infty, m\rangle$ and $\langle h_k, m\rangle = \langle h_\infty, m\rangle$. But since we also have $m >_{\mathrm{op}} m_k = \mathrm{lm}\,(g_k - h_k)$, then, we have $\langle g_k, m\rangle = \langle h_k, m\rangle$ and thus $\langle g_\infty, m\rangle = \langle h_\infty, m\rangle$. Since the monomial m was arbitrary, we have $g_\infty = h_\infty$. Letting $\ell = g_\infty = h_\infty$, we thus have:



Hence, $\to$ is infinitary confluent. ☐

As recalled in the beginning of the proof, from [3, Theorem 4.1.3], the rewriting relation $\to$ induced by a monomial order $<$ and a set G of formal power series is τ_δ-confluent if and only if G is a standard basis relative to $<$ of the ideal it generates. Hence, we get the following corollary, which is in fact what was showed in the proof of Theorem 4.2.2.

Corollary 4.2.3. *A set $G \subseteq \mathbb{K}[[x_1, \cdots, x_n]]$ is a standard basis relative to a monomial order $<$ that is compatible with the degree of the ideal it generates if and only if the rewriting relation induced by G and $<$ is infinitary confluent.*

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