Topological rewriting theory

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Contents

1 Topological preliminaries

Let (X, τ) be a topological space.

1.1 Separation axioms

Definition 1.1 : Basic separation notions

1. Two subsets of X are said to be **topologically indistinguishable** if they have exactly the same neighbourhoods.
2. Two subsets of X are said to be **separated** if each of them has neighbourhood that is not a neighbourhood of the other.
3. Two subsets of X are said to be **separated by (closed) neighbourhoods** if they have disjoint (closed) neighbourhoods.
4. Two subsets A, B of X are said to be **(resp. precisely) separated by a continuous function** if there exists a continuous function $f : X \to \mathbb{R}$ such that $A \subseteq f^{-1}(\{0\})$ (resp. $A = f^{-1}(\{0\})$) and $B \subseteq f^{-1}(\{1\})$ (resp. $B = f^{-1}(\{1\})$).

Definition 1.2 : Separation axioms

1. X is **Kolmogorov** (or T_0) if any two distinct points in X are topologically indistinguishable.
2. X is **Fréchet** (or T_1) if any two distinct points in X are separated. Equivalently, every single-point set is a closed set.
3. X is **Hausdorff** (or T_2) if any two distinct points in X are separated by neighbourhoods.
4. X is **Urysohn** (or $T_{2^1/_2}$) if any two distinct points in X are separated by closed neighbourhoods.
5. X is **completely Hausdorff** (or completely T_2) if any two distinct points in X are separated by a continuous function.
6. X is **regular Hausdorff** (or T_3) if it is both Kolmogorov and regular.
7. X is **Tychonoff** (or completely T_3) if it is both Kolmogorov and completely regular.
8. X is **normal Hausdorff** (or T_4) if it is both Fréchet and normal.
9. X is **completely normal Hausdorff** (or T_5) if it is both Fréchet and completely normal.
10. X is **perfectly normal Hausdorff** (or T_6) if it both Kolmogorov and perfectly normal.

Definition 1.3 : Normality axioms

1. X is **normal** if any two disjoint closed subsets of X are separated by neighbourhoods. Equivalently, if they are separated by a continuous function.
2. X is **completely normal** if any two separated sets are separated by neighbourhoods.

3. X is **perfectly normal** if any two disjoint closed sets are precisely separated by a continuous function.

Definition 1.4 : Regularity axioms

1. X is **symmetric** (or R_0) if any two topologically distinguishable points in X are separated.
2. X is **preregular** (or R_1) if any two topologically distinguishable points in X are separated by neighbourhoods.
3. X is **regular** if for any point x and closed set F in X such that $x \notin F$, F and $\{x\}$ are separated by neighbourhoods.
4. X is **completely regular** if for any point x and closed set F in X such that $x \notin F$, F and $\{x\}$ are separated by a continuous function.
5. X is **normal regular** if it is both symmetric and normal.

Proposition 1.5 : Relations between separation axioms

1. Perfectly normal Hausdorff $\Rightarrow$ completely normal Hausdorff $\Rightarrow$ normal Hausdorff $\Rightarrow$ Tychonoff $\Rightarrow$ completely Hausdorff $\Rightarrow$ Urysohn $\Rightarrow$ Hausdorff $\Rightarrow$ Fréchet $\Rightarrow$ Kolmogorov.
2. Normal Hausdorff $\Rightarrow$ normal regular $\Rightarrow$ completely regular $\Rightarrow$ regular $\Rightarrow$ preregular $\Rightarrow$ symmetric.
3. Tychonoff $\Rightarrow$ Regular Hausdorff $\Rightarrow$ Urysohn.
4. Perfectly normal $\Rightarrow$ completely normal $\Rightarrow$ normal.
5. Perfectly normal $\Rightarrow$ completely regular.
6. Fréchet $\Leftrightarrow$ Kolmogorov and symmetric.
7. Hausdorff $\Leftrightarrow$ Kolmogorov and preregular.

Proposition 1.6

Let (X, τ) be a topological space.
Then, X is Fréchet if, and only if, for all $A \subseteq X$, the intersection of all neighbourhoods of A is equal to A.

Proof. Suppose X is Fréchet. Let $A \subseteq X$ and write $I_A := \bigcap_{U \in \mathcal{N}(A)} U$. It is straightforward that $A \subseteq I_A$ by definition of neighbourhoods. If A is empty, then $\varnothing$ is a neighbourhood of A, and thus $I_A = \varnothing = A$. Otherwise, I_A is non-empty since it contains A. Let $x \in I_A$ which means that, for all $U \in \mathcal{N}(A)$, we have $x \in U$. By contradiction, suppose that $x \notin A$, then, since the space is Fréchet, for any $a \in A$, there exists an open neighbourhood of a that does not contain x, denote it by U_a. Therefore, $V := \bigcup_{a \in A} U_a$ does not contain x. However, it is an union of open sets, and therefore open, as well as a set that contains all points of A by construction; hence is a V is neighbourhood of A that does not contain x which contradicts the definition of x. Hence $A = I_A$.

Suppose now that for all $A \subseteq X$ we have $A = I_A$. Let $x, y \in X$ distinct points. Then, $I_{\{x\}} = \{x\}$ and $I_{\{y\}} = \{y\}$. By contradiction, if y is not seperated from x, *i.e.* there are no neighbourhood of

x that does not contain y, then $y \in I_{\{x\}}$ hence $y = x$, a contradiction. Same for x. Therefore, x and y are seperated and thus the space X is Fréchet. □

1.2 Countability axioms

Definition 1.7 : Countability axioms

1. X is **sequential** if a subset A of X is open if every sequence converging to a point in A is eventually in A.
2. X is **first-countable** if every point has a countable fundamental system of neighbourhoods.
3. X is **second-countable** if the topology has a countable base.
4. X is **separable** if there exists a countable dense subset.
5. X is **Lindelöf** if every open cover has a countable subcover.
6. X is **σ-compact** if there exists a countable cover by compact spaces.
7. X is **paracompact** if every open cover has an open refinement that is locally finite.

Proposition 1.8

1. Second-countable $\Rightarrow$ first-countable $\Rightarrow$ sequential.
2. Second-countable $\Rightarrow$ separable and Lindelöf.
3. σ-compact $\Rightarrow$ Lindelöf.
4. Regular and Lindelöf $\Rightarrow$ paracompact.
5. Hausdorff and paracompact $\Rightarrow$ normal.

1.3 Metric spaces

Proposition 1.9

Metrisable spaces (hence, metric spaces and discrete spaces) are:

- *perfectly normal Hausdorff*,
- *first-countable*,
- *paracompact*.

Proposition 1.10

For a metrisable space:

$$\text{second-countable} \Leftrightarrow \text{separable} \Leftrightarrow \text{Lindelöf}.$$

1.4 Differential manifolds

Proposition 1.11

Differential manifolds are:

- *second-countable* and *Hausdorff* by definition,
- *paracompact*,
- *perfectly normal Hausdorff*.

1.5 Uniform spaces

Proposition 1.12

Uniformisable spaces are:

- *completely regular*.

Proposition 1.13

For a uniformisable space:

$$\text{Tychonoff} \Leftrightarrow \text{Hausdorff} \Leftrightarrow \text{Kolmogorov}.$$

Proposition 1.14

Every metric space is completely uniformisable.
Every topological group is a uniformisable space.
Every regular paracompact space (in particular, every Hausdorff paracompact space) is completely uniformisable.

2 Basic definitions of abstract topological rewriting theory

2.1 Classical abstract rewriting theory

Definition 2.1 : Abstract rewriting system

A **(classical) abstract rewriting system** consists of the data of an ordered pair $(X, \longrightarrow)$ where X is a set and $\longrightarrow$ is a binary relation on X.

Notations for an abstract rewriting system $(X, \longrightarrow)$:

- $\xrightarrow{0} := \Delta X := \left\{(x,x) \in X^2\right\}$: the identity relation on X.
- $\xrightarrow{i+1} := \xrightarrow{i} \circ \longrightarrow = \left\{(x,y) \in X^2 \,\middle|\, \exists z \in X, x \xrightarrow{i} z \longrightarrow y\right\}$: the $(i+1)$-fold composition for $i \in \mathbb{N}$.
- $\xrightarrow{+} := \bigcup_{i=1}^{\infty} \xrightarrow{i}$: the transitive closure of $\longrightarrow$.
- $\xrightarrow{\star} := \xrightarrow{+} \cup \xrightarrow{0}$: the reflexive transitive closure of $\longrightarrow$.
- $\xrightarrow{=} := \longrightarrow \cup \xrightarrow{0}$: the reflexive closure of $\longrightarrow$.
- $\xrightarrow{-1} := \left\{(y,x) \in X^2 \mid x \longrightarrow y\right\}$: the inverse relation of $\longrightarrow$.

- $\longleftarrow := \overset{-1}{\longrightarrow}$: the inverse relation of $\longrightarrow$.
- $\longleftrightarrow := \longrightarrow \cup \longleftarrow$: the symmetric closure of $\longrightarrow$.
- $\overset{+}{\longleftrightarrow} := (\longleftrightarrow)^{+}$: the transitive symmetric closure of $\longrightarrow$.
- $\overset{\star}{\longleftrightarrow} := (\longleftrightarrow)^{\star}$: the reflexive transitive symmetric closure of $\longrightarrow$, *i.e.* the equivalence relation generated by $\longrightarrow$.

Definition 2.2 : Confluence definitions

Let $\mathcal{X} := (X, \longrightarrow)$ be an abstract rewriting system.

1. $\mathcal{X}$ is **Church-Rosser** if $\overset{\star}{\longleftrightarrow} \; \overset{\star}{\longrightarrow} \circ \overset{\star}{\longleftarrow}$.
2. $\mathcal{X}$ is **confluent** if $\overset{\star}{\longleftarrow} \circ \overset{\star}{\longrightarrow} \subseteq \overset{\star}{\longrightarrow} \circ \overset{\star}{\longleftarrow}$.
3. $\mathcal{X}$ is **semi-confluent** if $\longleftarrow \circ \overset{\star}{\longrightarrow} \subseteq \overset{\star}{\longrightarrow} \circ \overset{\star}{\longleftarrow}$.
4. $\mathcal{X}$ is **locally confluent** if $\longleftarrow \circ \longrightarrow \subseteq \overset{\star}{\longrightarrow} \circ \overset{\star}{\longleftarrow}$.
5. $\mathcal{X}$ is **subcommutative** if $\longleftarrow \circ \longrightarrow \subseteq \overset{=}{\longrightarrow} \circ \overset{=}{\longleftarrow}$.
6. $\mathcal{X}$ has the **diamond property** if $\longleftarrow \circ \longrightarrow \subseteq \longrightarrow \circ \longleftarrow$.

Proposition 2.3

For any abstract rewriting system $\mathcal{X} := (X, \longrightarrow)$ (write $\mathcal{X}^\star := (X, \overset{\star}{\longrightarrow})$), the following are equivalent:

(i) $\mathcal{X}$ is confluent.

(ii) $\mathcal{X}$ is semi-confluent.

(iii) $\mathcal{X}$ is Church-Rosser.

(iv) $\mathcal{X}^\star$ is locally confluent.

(v) $\mathcal{X}^\star$ is subcommutative.

(vi) $\mathcal{X}^\star$ has the diamond property.

Definition 2.4 : Normalisation

Let $\mathcal{X} := (X, \longrightarrow)$ be an abstract rewriting system.

1. $n \in X$ is a **normal form** if $\{y \in X \mid n \longrightarrow y\} = \varnothing$. Denote by $\mathrm{NF}(\mathcal{X})$ the set of normal forms of $\mathcal{X}$.
2. $\mathcal{X}$ is **normalising** if, for all $x \in X$,there exists a normal form $n \in \mathrm{NF}(\mathcal{X})$ such that $x \overset{\star}{\longrightarrow} n$.
3. $\mathcal{X}$ is **terminating** if for every $x \in X$, the reduction sequences starting from x are finite.
4. $\mathcal{X}$ has the **normal form property** if, for all $x \in X$ and $n \in \mathrm{NF}(\mathcal{X})$, $x \overset{\star}{\longleftrightarrow} n \Rightarrow x \overset{\star}{\longrightarrow} n$.
5. $\mathcal{X}$ has the **unique normal form property** if, for all $n_0, n_1 \in \mathrm{NF}(\mathcal{X})$, $n_0 \overset{\star}{\longleftrightarrow} n_1 \Rightarrow n_0 = n_1$.

Theorem 2.5 : (Newman's lemma)

Let $\mathcal{X} := (X, \longrightarrow)$ be an abstract rewriting system.
If $\mathcal{X}$ is terminating and locally confluent, then it is confluent.

Theorem 2.6

Let $\mathcal{X} := (X, \longrightarrow)$ be an abstract rewriting system.
Then we have the following properties:

1. If $\mathcal{X}$ is confluent, then it has the unique normal form property.
2. The normal form property and the unique normal form property are equivalent in $\mathcal{X}$.
3. If $\mathcal{X}$ is normalising and has the unique normal form property, then it is Church-Rosser.
4. If $\mathcal{X}$ is subcommutative, then it is confluent.

2.2 Abstract topological rewriting theory

Definition 2.7 : Abstract topological rewriting system

An **(abstract) topological rewriting system** $\mathcal{X}$ consists of the data of a triple $(X, \tau, \longrightarrow)$ where (X, τ) is a topological space and $\longrightarrow$ is a binary relation on X.

For a topological rewriting system $\mathcal{X} := (X, \tau, \longrightarrow)$, we write $\mathrm{NF}(\mathcal{X})$ for the set of normal forms for the classical rewriting system $(X, \longrightarrow)$.

Definition 2.8 : Topological rewriting relation

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
The closure of $\overset{\star}{\longrightarrow}$ in the product space X^2 endowed with the topology $\tau_{\mathrm{dis}}^X \times \tau$ is called the **topological rewriting relation of** $\mathcal{X}$ and is denoted $\longrightarrow\!\!\!\!\circledast$.
We define the **limit rewriting relation** as: $\underset{\lim}{\longrightarrow} := \longrightarrow\!\!\!\!\circledast \setminus \overset{\star}{\longrightarrow}$.
Write $\overset{\star}{\longrightarrow\!\!\!\!\circledast}$ (resp. $\circledast\!\!\!\!\overset{\star}{\longleftrightarrow}\!\!\!\!\circledast$) the transitive closure of (resp. the equivalence relation generated by) $\longrightarrow\!\!\!\!\circledast$.

Proposition 2.9

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
Then, for all $a, b \in X$, we have $a \longrightarrow\!\!\!\!\circledast\, b$ if, and only if, for every neighbourhood U of b for the topology τ, there exists $c \in U$ such that $a \overset{\star}{\longrightarrow} c$.

Proposition 2.10

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
Then, for all $a, b \in X$ if, and only if, there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in X such that $c_0 = a$, $a \overset{\star}{\longrightarrow} c_n$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} c_n = b$.

Corollary 2.11

For all $a, b \in X$, if $a \overset{\star}{\longrightarrow} b$ then $a \longrightarrow\!\!\!\!\circledast\, b$, *i.e.* $\overset{\star}{\longrightarrow} \subseteq \longrightarrow\!\!\!\!\circledast$ as relations on X.

This is because b is contained in any of its neighbourhoods.

Definition 2.12 : Discrete rewriting

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
The system $\mathcal{X}$ has **discrete rewriting** if for all $a, b \in X$, having $a \mathrel{-\!\!\oplus} b$ implies $a \xrightarrow{\star} b$.
This is equivalent to asserting any of the following statements:

- $\mathrel{-\!\!\oplus} \; = \; \xrightarrow{\star}$ as relations on X,
- $\xrightarrow{\star}$ is topologically closed in X^2 for the product topology $\tau^X_{\text{dis}} \times \tau$,
- $\underset{\lim}{\longrightarrow} = \varnothing$.

Proposition 2.13

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
If $\tau = \tau^X_{\text{dis}}$ is the discrete topology on X, then the system $\mathcal{X}$ has discrete rewriting.

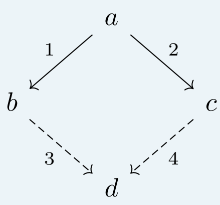
The converse implication is not verified as shown in the following example:

Example 2.14 : (Discrete rewriting without discrete topology)

Consider the Sierpiński space $X := \{0, 1\}$ equipped with the topology $\tau = \{\varnothing, \{1\}, X\}$. Consider now the relation on X given by $1 \longrightarrow 0$. Then $\xrightarrow{\star} = \{(0,0), (1,0), (1,1)\}$. Hence, the system has discrete rewriting if and only if $(0,1)$ is not adherent to $\xrightarrow{\star}$ for the product topology $\tau^X_{\text{dis}} \times \tau$. But since $\{1\}$ is open in (X, τ), the set $\{0\} \times \{1\}$ is a neighbourhood of $(0,1)$ for the topology $\tau^X_{\text{dis}} \times \tau$ and it does not intersect $\xrightarrow{\star}$, hence $\xrightarrow{\star}$ is closed for $\tau^X_{\text{dis}} \times \tau$ and, therefore, the system has discrete rewriting.

Definition 2.15 : Branching closure

Let A, B, C, D be sets.
Let $\underset{1}{\to} \subseteq A \times B$, $\underset{2}{\to} \subseteq A \times C$, $\underset{3}{\to} \subseteq B \times D$, $\underset{4}{\to} \subseteq C \times D$ be binary relations.
We say that the **branching of** $\underset{1}{\to}$ **and** $\underset{2}{\to}$ **closes with respect to** $\underset{3}{\to}$ **and** $\underset{4}{\to}$ if for all $a \in A$, $b \in B$ and $c \in C$ such that $b \underset{1}{\leftarrow} a \underset{2}{\to} c$, there exists $d \in D$ such that $b \underset{3}{\to} d \underset{4}{\leftarrow} c$.
In diagrams:



Consider the case where $B = C$. When $\underset{1}{\to} = \underset{2}{\to} =: \longrightarrow$ (resp. $\underset{3}{\to} = \underset{4}{\to} =: \longrightarrow$) we talk about **branching closure of** $\longrightarrow$ **with respect to** $\underset{3}{\to}$ **and** $\underset{4}{\to}$ (resp. **branching closure of** $\underset{1}{\to}$ **and** $\underset{2}{\to}$ **with respect to** $\longrightarrow$).

Lemma 2.16

Let A, B, C, D be sets. Let $\underset{1}{\to}, \underset{1}{\rightsquigarrow} \subseteq A \times B$, $\underset{2}{\to}, \underset{2}{\rightsquigarrow} \subseteq A \times C$, $\underset{3}{\to}, \underset{3}{\rightsquigarrow} \subseteq B \times D$, $\underset{4}{\to}, \underset{4}{\rightsquigarrow} \subseteq C \times D$ be binary relations.
Assume that:

- $\underset{1}{\rightsquigarrow} \subseteq \underset{1}{\to}$,

- $\underset{2}{\rightsquigarrow} \subseteq \underset{2}{\rightarrow}$,
- $\underset{3}{\rightarrow} \subseteq \underset{3}{\rightsquigarrow}$,
- $\underset{4}{\rightarrow} \subseteq \underset{3}{\rightsquigarrow}$,

Then, if the branching of $\underset{1}{\rightarrow}$ and $\underset{2}{\rightarrow}$ closes with respect to $\underset{3}{\rightarrow}$ and $\underset{4}{\rightarrow}$, it follows that the branching of $\underset{1}{\rightsquigarrow}$ and $\underset{2}{\rightsquigarrow}$ closes with respect to $\underset{3}{\rightsquigarrow}$ and $\underset{4}{\rightsquigarrow}$.

Lemma 2.17

Let X and D be sets. Let $\overset{\star}{\leftrightarrow}$ be an equivalence relation on X and $\underset{3}{\rightarrow}$, $\underset{4}{\rightarrow}$ binary relations from X to D. Let Y be a subset of X and write $\overset{\star}{\leftrightsquigarrow}$ the corestriction of $\overset{\star}{\leftrightarrow}$ to Y. Then, the branching of $\overset{\star}{\leftrightarrow}$ and $\overset{\star}{\leftrightsquigarrow}$ closes with respect to $\underset{3}{\rightarrow}$ and $\underset{4}{\rightarrow}$ if, and only if, for all $x \in X$ and $y \in Y$ such that $x \overset{\star}{\leftrightsquigarrow} y$, there exists $d \in D$ with $x \underset{3}{\rightarrow} d \underset{4}{\leftarrow} y$.

Corollary 2.18

Let X and D be sets. Let $\longrightarrow$ be a binary relation from X to D.
Then, for all $x \in X$ there exists $d \in D$ such that $x \longrightarrow d$ if, and only if, the branching of $=$ (equality on X) closes with respect to $\longrightarrow$.

Definition 2.19 : Local, finitary and infinitary confluences

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system. Let $\rightsquigarrow$ be a binary relation on X.
We say that **the system $\mathcal{X}$ is locally confluent with respect to** $\rightsquigarrow$ if the branching of $\longrightarrow$ closes with respect to $\rightsquigarrow$.
We say that **the system $\mathcal{X}$ is finitary confluent with respect to** $\rightsquigarrow$ if the branching of $\overset{\star}{\longrightarrow}$ closes with respect to $\rightsquigarrow$.
We say that **the system $\mathcal{X}$ is infinitary confluent with respect to** $\rightsquigarrow$ if the branching of $-\!\!\circledast$ closes with respect to $\rightsquigarrow$.

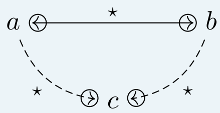
Since for all topological rewriting systems we have $\longrightarrow \subseteq \overset{\star}{\longrightarrow} \subseteq -\!\!\circledast$, then, by Lemma 2.16, if $\rightsquigarrow$ is a binary relation on X:

- finitary confluence with respect to $\rightsquigarrow$ implies local confluence with respect to $\rightsquigarrow$,
- infinitary confluence w.r.t $\rightsquigarrow$ implies finitary confluence w.r.t $\rightsquigarrow$.

Definition 2.20 : Topological Church-Rosser

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{-\!\!\circledast}$. Denote $\overset{\star}{\rightsquigarrow}$ and $\overset{\star}{\leftrightsquigarrow}$ the reflexive transitive closure of $\rightsquigarrow$ and the equivalence relation generated by $\rightsquigarrow$ respectively.
We say that the system $\mathcal{X}$ has the **Church-Rosser property with respect to** $\rightsquigarrow$ if the branching of $\overset{\star}{\leftrightsquigarrow}$ closes with respect to $\overset{\star}{\rightsquigarrow}$. That is to say, for every $a, b \in X$ such that $a \overset{\star}{\leftrightsquigarrow} b$, there exists $c \in X$ such that $a \overset{\star}{\rightsquigarrow} c \overset{\star}{\leftsquigarrow} b$.
We say that the system $\mathcal{X}$ has the **topological Church-Rosser property** if is has the Church-Rosser property with respect to $-\!\!\circledast$.
This is equivalent to asserting that, for all $a, b \in X$ such that $a \circledast\!\!\overset{\star}{-\!-}\!\!\circledast b$, there exists $c \in X$ such that $a \overset{\star}{-\!\!\circledast} c \overset{\star}{\circledast\!\!-} b$.

In terms of diagrams:



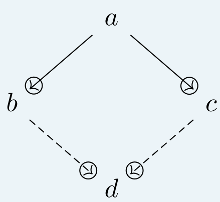
This is the classical Church-Rosser property of the classical rewriting system $(X, \multimap)$.

Remark 2.21

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\multimap}$. Let $\underset{\text{sub}}{\rightsquigarrow}$ be a relation on X such that $\longrightarrow \subseteq \underset{\text{sub}}{\rightsquigarrow} \subseteq \rightsquigarrow$. Assume that $\overset{\star}{\leftrightsquigarrow} \subseteq \underset{\text{sub}}{\overset{\star}{\leftrightsquigarrow}}$ (*i.e.* $\rightsquigarrow$ and $\underset{\text{sub}}{\rightsquigarrow}$ generate the same equivalence relation). Then, if the system $\mathcal{X}$ verifies the Church-Rosser property with respect to $\underset{\text{sub}}{\rightsquigarrow}$, then it also has the Church-Rosser property with respect to $\rightsquigarrow$.

Definition 2.22 : Infinitary confluence

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
We say that the system $\mathcal{X}$ is **infinitary confluent** if the system is infinitary confluent with respect to $\multimap$. That is to say, for any $a, b, c \in X$ such that $b \multimapinv a \multimap c$, there exists $d \in X$ that satisfies $b \multimap d \multimapinv c$.
In terms of diagrams:



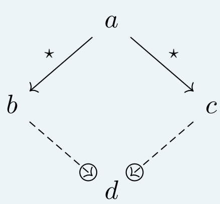
This is exactly the diamond property of the classical rewriting system $(X, \multimap)$.

It follows that infinitary confluence implies the topological Church-Rosser property from any system (since in classical rewriting theory, it is can be shown by induction that the diamond property implies confluence which is equivalent to the classical Church-Rosser property).

Moreover, if $\multimap$ is transitive, then the diamond property of $(X, \multimap)$ is equivalent to its confluence. Hence, under the assumption that $\multimap$ is transitive, infinitary confluence and the topological Church-Rosser property are equivalent.

Definition 2.23 : Topological confluence

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
We say that the system $\mathcal{X}$ is **topologically confluent** if the system is finitary confluent with respect to $\multimap$, that is to say: for any $a, b, c \in X$ such that $b \overset{\star}{\longleftarrow} a \overset{\star}{\longrightarrow} c$ there exists $d \in X$ that satisfies $b \multimap d \multimapinv c$.
In terms of diagrams:



We obtain the following strength implications (recall that classical confluence is nothing other than finitary confluence with respect to $\overset{\star}{\longrightarrow}$):

Proposition 2.24

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
If the system $\mathcal{X}$ is:

- *classically confluent*, or,
- *infinitary confluent*,

then it is also *topologically confluent*.

The converse implications are false in general, we will give counter-examples later.

Remark 2.25

If the system $\mathcal{X}$ has discrete rewriting, it is trivial to see that the notions of classical confluence, topological confluence and infinitary confluence are equivalent.

Proposition 2.26

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system. Let $\rightsquigarrow$ be a binary relation on X.

- Suppose (X, τ) is Fréchet and $\rightsquigarrow \subseteq \overset{\star}{\multimap}$. Then, for any normal form $n \in \mathrm{NF}(\mathcal{X})$, we have that, for all $a \in X$, if $n \rightsquigarrow a$ then $n = a$. It follows that if $\rightsquigarrow$ is reflexive, then $\rightsquigarrow$ restricted to $\mathrm{NF}(\mathcal{X})$ is exactly the equality on $\mathrm{NF}(\mathcal{X})$.
- Suppose $\longrightarrow$ is anti-reflexive and $\longrightarrow \subseteq \rightsquigarrow$. Then, if $a \in X$ is such that for all $b \in X$ with $a \rightsquigarrow b$ implies $a = b$, then $a \in \mathrm{NF}(\mathcal{X})$.

Proof. Assume (X, τ) is Fréchet. Let $n \in \mathrm{NF}(\mathcal{X})$ *i.e.* $\{a \in X \mid n \longrightarrow a\} = \varnothing$. Let $a \in X$ such that $n \multimap a$. Then, for all neighbourhood U of a there exists $b \in U$ such that $n \overset{\star}{\longrightarrow} b$. But, since n is a normal form, $n = b$. Hence, $n \in \bigcap_{U \in \mathcal{N}(a)} U$. But that latter set is equal to $\{a\}$ because the space is Fréchet. Then, $n = a$. Now since $\rightsquigarrow$ is by assumption a subrelation of $\overset{\star}{\multimap}$, if $n \rightsquigarrow a$ then we have $n \overset{\star}{\multimap} a$ and hence $n = a$ by applying inductively the previous discussion.

Assume now $\longrightarrow$ anti-reflexive. Let $a \in X$ such that for all $b \in X$ with $a \rightsquigarrow b$ implies $a = b$. Assume there exists $b \in X$ such that $a \longrightarrow b$. On one hand, since $\longrightarrow$ is by assumption a subrelation of $\rightsquigarrow$, this implies $a \rightsquigarrow b$ and hence $a = b$ by assumption. On the other hand, by anti-reflexivity of $\longrightarrow$, this also implies $a \neq b$, a contradiction. Hence $\{b \in X \mid a \longrightarrow b\} = \varnothing$, that is to say, $a \in \mathrm{NF}(\mathcal{X})$. $\square$

Definition 2.27 : Topological normal forms properties

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system. Let $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\multimap}$. Denote $\overset{\star}{\rightsquigarrow}$ and $\overset{\star}{\leftrightsquigarrow}$ the reflexive transitive closure of $\rightsquigarrow$ and the equivalence relation generated by $\rightsquigarrow$ respectively.

- The system $\mathcal{X}$ has the **property of the normal form (NF) with respect to** $\rightsquigarrow$ if the branching of $\overset{\star}{\leftrightsquigarrow}$ (corestricted to $\mathrm{NF}(\mathcal{X})$) closes with respect to $\overset{\star}{\rightsquigarrow}$ and $=$ (equality on $\mathrm{NF}(\mathcal{X})$), that is to say:

$$\forall a \in X, \quad \forall n \in \mathrm{NF}(\mathcal{X}), \quad a \overset{\star}{\leftrightsquigarrow} n \;\Rightarrow\; a \overset{\star}{\rightsquigarrow} n.$$

- The system $\mathcal{X}$ has the **topological property of the normal form** (**TNF**) if it has the property of the normal form with respect to $\mathrel{-\!\circledast}$, *i.e.* :

$$\forall a \in X, \quad \forall n \in \mathrm{NF}(\mathcal{X}), \quad a \circledast\!\overset{\star}{-\!-}\!\circledast n \ \Rightarrow\ a \overset{\star}{\mathrel{-\!\circledast}} n.$$

- The system $\mathcal{X}$ has the **property of the unique normal form (UN) with respect to** $\rightsquigarrow$ if the branching of $\overset{\star}{\leftrightsquigarrow}$ (restricted and corestricted to $\mathrm{NF}(\mathcal{X})$) closes with respect to $=$ (equality on $\mathrm{NF}(\mathcal{X})$), that is to say:

$$\forall n, n' \in \mathrm{NF}(\mathcal{X}), \quad n \overset{\star}{\leftrightsquigarrow} n' \ \Rightarrow\ n = n'.$$

- The system $\mathcal{X}$ has the **topological property of the unique normal form** (**TUN**) if the property of the unique normal form with respect to $\mathrel{-\!\circledast}$, *i.e.* :

$$\forall n, n' \in \mathrm{NF}(\mathcal{X}), \quad n \circledast\!\overset{\star}{-\!-}\!\circledast n' \ \Rightarrow\ n = n'.$$

Remark 2.28

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\mathrel{-\!\circledast}}$. Let $\underset{\mathrm{sub}}{\rightsquigarrow}$ be a relation on X such that $\longrightarrow \subseteq \underset{\mathrm{sub}}{\rightsquigarrow} \subseteq \rightsquigarrow$.
By Lemma 2.16, if the system $\mathcal{X}$ is UN with respect to $\rightsquigarrow$, then it is also UN with respect to $\underset{\mathrm{sub}}{\rightsquigarrow}$. In particular, if the system is TUN, then it is also UN with respect to $\rightsquigarrow$.
Now, assume that $\overset{\star}{\leftrightsquigarrow} \subseteq \underset{\mathrm{sub}}{\overset{\star}{\leftrightsquigarrow}}$ (*i.e.* $\rightsquigarrow$ and $\underset{\mathrm{sub}}{\rightsquigarrow}$ generate the same equivalence relation) then:

- the system is UN with respect to $\rightsquigarrow$ if and only if it is UN with respect to $\underset{\mathrm{sub}}{\rightsquigarrow}$,
- if the system is NF with respect to $\underset{\mathrm{sub}}{\rightsquigarrow}$, then it is also NF with respect to $\rightsquigarrow$.

Proposition 2.29

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\mathrel{-\!\circledast}}$.
Assume that the space (X, τ) is Fréchet.
Then, if the system $\mathcal{X}$ is NF with respect to $\rightsquigarrow$, it is also UN with respect to $\rightsquigarrow$.

Proof. If we denote $\underset{1}{\overset{\star}{\leftrightsquigarrow}}$ the corestriction of $\overset{\star}{\leftrightsquigarrow}$ to $\mathrm{NF}(\mathcal{X})$ and $\underset{2}{\overset{\star}{\leftrightsquigarrow}}$ the restriction of $\underset{1}{\overset{\star}{\leftrightsquigarrow}}$ to $\mathrm{NF}(\mathcal{X})$, it is clear that $\underset{2}{\overset{\star}{\leftrightsquigarrow}}$ is a subrelation of $\underset{1}{\overset{\star}{\leftrightsquigarrow}}$.

Now, because by hypothesis the space is Fréchet, we obtain by Proposition 2.26 that $\overset{\star}{\rightsquigarrow}$ restricted to $\mathrm{NF}(\mathcal{X})$ is actually the equality on $\mathrm{NF}(\mathcal{X})$.

We conclude therefore by Lemma 2.16 that if the system is NF with respect to $\rightsquigarrow$ it is also UN with respect to that same relation. □

Proposition 2.30

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\mathrel{-\!\circledast}}$.
Assume that the space (X, τ) is Fréchet.
Then, if the system $\mathcal{X}$ verifies the Church-Rosser property with respect to $\rightsquigarrow$, it is NF with respect to $\rightsquigarrow$.

Proof. If we denote $\underset{1}{\overset{\star}{\leftrightsquigarrow}}$ the corestriction of $\overset{\star}{\leftrightsquigarrow}$ to $\mathrm{NF}(\mathcal{X})$, it is clear that $\underset{1}{\overset{\star}{\leftrightsquigarrow}}$ is a subrelation of $\overset{\star}{\leftrightsquigarrow}$. Morever, because by hypothesis the space is Fréchet, we deduce from Proposition 2.26 that $\overset{\star}{\rightsquigarrow}$ restricted to $\mathrm{NF}(\mathcal{X})$ is exactly the equality on $\mathrm{NF}(\mathcal{X})$. Therefore, NF with respect to $\rightsquigarrow$ is a branching closure property weaker than the Church-Rosser property with respect to $\rightsquigarrow$ according to Lemma 2.16. □

Corollary 2.31

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
Assume the space (X, τ) is Fréchet. Then:

$$\text{infinitary conf} \;\Rightarrow\; \text{topo. C-R} \;\Rightarrow\; \text{TNF} \;\Rightarrow\; \text{TUN}.$$

Corollary 2.32

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\longrightarrow}\!\!\circledcirc$. Let $\underset{\text{sub}}{\rightsquigarrow}$ be a relation on X such that $\longrightarrow \subseteq \underset{\text{sub}}{\rightsquigarrow} \subseteq \rightsquigarrow$.
Assume the space (X, τ) is Fréchet and that $\rightsquigarrow$ and $\underset{\text{sub}}{\rightsquigarrow}$ generate the same equivalence relation (*i.e.* we have $\overset{\star}{\leftrightsquigarrow} \subseteq \underset{\text{sub}}{\overset{\star}{\leftrightsquigarrow}}$). Then:

$$\text{C-R w.r.t } \underset{\text{sub}}{\rightsquigarrow} \;\Rightarrow\; \left(\text{C-R w.r.t } \rightsquigarrow \;\wedge\; \text{NF w.r.t } \underset{\text{sub}}{\rightsquigarrow}\right)$$

$$\left(\text{C-R w.r.t } \rightsquigarrow \;\vee\; \text{NF w.r.t } \underset{\text{sub}}{\rightsquigarrow}\right) \;\Rightarrow\; \text{NF w.r.t } \rightsquigarrow$$

$$\text{NF w.r.t } \rightsquigarrow \;\Rightarrow\; \text{UN w.r.t } \rightsquigarrow$$

$$\text{UN w.r.t } \rightsquigarrow \;\Leftrightarrow\; \text{UN w.r.t } \underset{\text{sub}}{\rightsquigarrow}$$

Example 2.33 : (Counter-example "topological confluence $\Rightarrow$ infinitary confluence")

Consider the set $(\omega + 1) := \mathbb{N} \cup \{\infty\}$. This is the successor ordinal to ω, the first infinite ordinal. It can be equipped with the order topology. Concretely, this means that for every $n \in \mathbb{N}$, $\{n\}$ is open but a fundamental system of neighbourhoods of ∞ in that topological space is given by the family of intervals $[\![a \,..\, \infty]\!] := \{n \in \mathbb{N} \mid a \leqslant n\}$ where a lives in $\mathbb{N}$. It follows that the space is Hausdorff (hence, Fréchet *a fortiori*).
Consider the set $X := (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$ equipped with the product topology τ of the order topology on each factor. The space X is thus also Hausdorff.
Now, construct the topological rewriting system on (X, τ) by defining the following relation on X:

$$(n, m) \longrightarrow (n+1, m) \qquad (n, m) \longrightarrow (n, m+1) \qquad \forall n, m \in \mathbb{N}.$$

Hence, it follows that $(n_1, m_1) \overset{\star}{\longrightarrow} (n_2, m_2)$ if and only if $n_1, n_2, m_1, m_2 \in \mathbb{N}$ and $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$. Therefore, the system is confluent. Indeed, for all $n, m, n_1, m_1, n_2, m_2 \in \mathbb{N}$ such that:

$$\begin{array}{ccc} & (n, m) & \\ {}^{\star}\swarrow & & \searrow^{\star} \\ (n_1, m_1) & & (n_2, m_2) \end{array}$$

we have:

$$(n_1, m_1) \overset{\star}{\dashrightarrow} (\max\{n_1, n_2\}, \max\{m_1, m_2\}) \overset{\star}{\dashleftarrow} (n_2, m_2)$$

Hence, by confluence, it follows that the system is topologically confluent.
However, we have:

$$\begin{array}{ccccccccc} & & (1,0) & \longrightarrow & (2,0) & \longrightarrow & \cdots & \longrightarrow\!\!\!\circledast & (\infty,0) \\ & \nearrow & & & & & & & \\ (0,0) & & & & & & & & \\ & \searrow & & & & & & & \\ & & (0,1) & \longrightarrow & (0,2) & \longrightarrow & \cdots & \longrightarrow\!\!\!\circledast & (0,\infty) \end{array}$$

And $(0, \infty)$ and $(\infty, 0)$ are distinct normal forms in the system. Hence, since the space is Fréchet, the system cannot be infinitary confluent, since that would imply the topological property of unique normal form.

Proposition 2.34

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\longrightarrow}\!\!\!\circledast$. Let $\underset{\text{sub}}{\rightsquigarrow}$ be a relation on X such that $\longrightarrow \subseteq \underset{\text{sub}}{\rightsquigarrow} \subseteq \rightsquigarrow$. Assume that $\rightsquigarrow$ and $\underset{\text{sub}}{\rightsquigarrow}$ generate the same equivalence relation (*i.e.* we have $\overset{\star}{\leftrightsquigarrow} \subseteq \underset{\text{sub}}{\overset{\star}{\leftrightsquigarrow}}$). Then the Church-Rosser property with respect to $\underset{\text{sub}}{\rightsquigarrow}$ implies that the branching of $\overset{\star}{\rightsquigarrow}$ closes with respect to $\underset{\text{sub}}{\overset{\star}{\rightsquigarrow}}$.

Proof. Assume that the system verifies the Church-Rosser property with respect to $\underset{\text{sub}}{\rightsquigarrow}$. Let $a, b, c \in X$ such that $b \overset{\star}{\leftsquigarrow} a \overset{\star}{\rightsquigarrow} c$. Then, $b \overset{\star}{\leftrightsquigarrow} c$. But since by hypothesis $\rightsquigarrow$ and $\underset{\text{sub}}{\rightsquigarrow}$ generate the same relation, we get $b \underset{\text{sub}}{\overset{\star}{\leftrightsquigarrow}} c$ and therefore, by Church-Rosser assumption, there exists $d \in X$ such that $b \underset{\text{sub}}{\overset{\star}{\rightsquigarrow}} d \underset{\text{sub}}{\overset{\star}{\leftsquigarrow}} c$. Thus the conclusion. □

Corollary 2.35

With the same hypotheses and notations, the Church-Rosser property with respect to $\underset{\text{sub}}{\rightsquigarrow}$ implies that the system is finitary confluent with respect to $\underset{\text{sub}}{\overset{\star}{\rightsquigarrow}}$ and, therefore, also with respect to $\overset{\star}{\rightsquigarrow}$.

Definition 2.36 : Topologically normalising

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\overset{=}{\longrightarrow} \subseteq \rightsquigarrow \subseteq \overset{\star}{\longrightarrow}\!\!\!\circledast$.
We say that the system $\mathcal{X}$ is **normalising with respect to** $\rightsquigarrow$ if the branching of the equality on X closes with respect to $\rightsquigarrow$ corestricted to normal forms, that is to say, if:

$$\forall a \in X, \quad \exists n \in \mathrm{NF}(\mathcal{X}), \quad a \rightsquigarrow n.$$

We say that the system $\mathcal{X}$ is **topologically normalising** (**TN**) if it is normalising with respect to $\overset{\star}{\longrightarrow}\!\!\!\!\circledast$, that is to say:

$$\forall a \in X, \quad \exists n \in \mathrm{NF}(\mathcal{X}), \quad a \overset{\star}{\twoheadrightarrow} n.$$

Lemma 2.37

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\twoheadrightarrow}$.
Suppose that the space (X, τ) is *Fréchet* and that the system $\mathcal{X}$ is normalising with respect to $\overset{\star}{\rightsquigarrow}$.
Then being UN with respect to $\rightsquigarrow$ is equivalent to the following statement:

$$\forall a \in X, \quad \forall n, n' \in \mathrm{NF}(\mathcal{X}), \quad n \overset{\star}{\leftsquigarrow} a \overset{\star}{\rightsquigarrow} n' \;\Rightarrow\; n = n'.$$

Proof. The left-to-right direction is trivial and does not require any of the hypotheses. Now for the other direction:

Let $n, n' \in \mathrm{NF}(\mathcal{X})$ such that $n \overset{\star}{\leftrightsquigarrow} n'$. Decompose it with elements $(a_1, \cdots, a_\ell) \in X^\ell$ such that:

$$n \leftrightsquigarrow a_1 \leftrightsquigarrow a_2 \leftrightsquigarrow \cdots \leftrightsquigarrow a_\ell \leftrightsquigarrow n'.$$

Now, since the space is Fréchet by hypothesis, we get that either $a_1 = n$ (resp. $a_\ell = n'$) or $n \leftsquigarrow a_1$ (resp. $a_\ell \rightsquigarrow n'$) by Proposition 2.26. Assume thus, by contradiction that $n \neq n'$ and therefore $a_1 \neq n$ and $a_\ell \neq n'$. Now, call a "valley" in the sequence $a_1 \leftrightsquigarrow a_2 \leftrightsquigarrow \cdots \leftrightsquigarrow a_\ell$ an index $i \in [\![2 \,..\, \ell - 1]\!]$ such that $a_{i-1} \rightsquigarrow a_i \leftsquigarrow a_{i+1}$. Show by induction on the number ν of valleys in the sequence that $n = n'$.

Base step: if there are no valleys, then the sequence is of the form:

$$n \leftsquigarrow a_1 \leftsquigarrow a_2 \leftsquigarrow \cdots \leftsquigarrow a_i \rightsquigarrow a_{i+1} \rightsquigarrow \cdots \rightsquigarrow a_\ell \rightsquigarrow n'.$$

Therefore, it suffices to use the assumption to conclude that $n = n'$ because we have $n \overset{\star}{\leftsquigarrow} a_i \overset{\star}{\rightsquigarrow} n'$.

Induction step: let $\nu \in \mathbb{N}$. Assume that, if there are up to ν valleys between n and n' then $n = n'$. Consider there are $\nu + 1$ valleys and let $i \in [\![2 \,..\, \ell - 1]\!]$ be any of the valleys, say the right-most for simplicity, *i.e.* we have:

$$a_{i-1} \rightsquigarrow a_i \leftsquigarrow a_{i+1} \leftrightsquigarrow a_{i+2} \leftrightsquigarrow \cdots \leftrightsquigarrow a_\ell \rightsquigarrow n'.$$

Since this is by construction the right-most valley, there exists $j \in [\![i + 2 \,..\, \ell]\!]$ such that:

$$a_{i-1} \rightsquigarrow a_i \leftsquigarrow a_{i+1} \leftsquigarrow a_{i+2} \leftsquigarrow \cdots \leftsquigarrow a_j \rightsquigarrow a_{j+1} \rightsquigarrow \cdots \rightsquigarrow a_\ell \rightsquigarrow n'.$$

Since the system $\mathcal{X}$ is normalising with respect to $\overset{\star}{\rightsquigarrow}$ there exists $n'' \in \mathrm{NF}(\mathcal{X})$ such that $a_i \overset{\star}{\rightsquigarrow} n''$. Hence, note how we obtain:

$$a_{i-1} \overset{\star}{\rightsquigarrow} n'' \overset{\star}{\leftsquigarrow} a_j \overset{\star}{\rightsquigarrow} n'.$$

But by assumption we conclude that $n' = n''$. Hence, the following sequence obtains by considering everything that is left to a_{i-1}:

$$n \leftsquigarrow a_1 \leftrightsquigarrow a_2 \leftrightsquigarrow \cdots \leftrightsquigarrow a_{i-1} \overset{\star}{\rightsquigarrow} n'$$

contains ν valleys and therefore, by induction hypothesis, we conclude that $n = n'$. □

The hypothesis that the system is normalising is required as shown by the following counter-example:

Example 2.38

Consider $X := \{n, a, c, b, n'\}$ a set of 5 distinct elements equipped with the discrete topology and the relation:

$$n \longleftarrow a \longrightarrow c \longleftarrow b \longrightarrow n' \qquad \text{and} \qquad c \longrightarrow c.$$

This gives a topological rewriting system $\mathcal{X}$.
It is clear that $\mathrm{NF}(\mathcal{X}) = \{n, n'\}$. Note how c never rewrites into n nor n': the system is not normalising with respect to $\overset{\star}{\longrightarrow}$.
But since $n \overset{\star}{\longleftrightarrow} n'$ and $n \neq n'$, then the system is not UN with respect to $\longrightarrow$.
However, if we denote, for $x \in X$, $\mathrm{NF}(x) := \left\{\alpha \in \mathrm{NF}(\mathcal{X}) \;\middle|\; x \overset{\star}{\longrightarrow} \alpha\right\}$, then we have:

$$\mathrm{NF}(n) = \{n\}, \quad \mathrm{NF}(a) = \{n\}, \quad \mathrm{NF}(c) = \varnothing, \quad \mathrm{NF}(b) = \{n'\}, \quad \mathrm{NF}(n') = \{n'\}.$$

Hence, for all $x \in X$ and $\alpha, \beta \in \mathrm{NF}(\mathcal{X})$ such that $\alpha \overset{\star}{\longleftarrow} x \overset{\star}{\longrightarrow} \beta$, we do indeed have $\alpha = \beta$.

Proposition 2.39

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \;\subseteq\; \rightsquigarrow \;\subseteq\; \overset{\star}{\longrightarrow}\!\!\!\!\bigcirc$.
Then, the system $\mathcal{X}$ is UN with respect to $\rightsquigarrow$ and normalising with respect to $\overset{\star}{\rightsquigarrow}$ if, and only if, it satisfies:

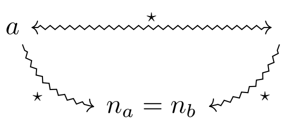
$$\forall a \in X, \quad \exists! n \in \mathrm{NF}(\mathcal{X}), \quad a \overset{\star}{\rightsquigarrow} n.$$

Proof. The left-to-right direction is immediate. The other direction is obtained with Lemma 2.37. ☐

Proposition 2.40

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \;\subseteq\; \rightsquigarrow \;\subseteq\; \overset{\star}{\longrightarrow}\!\!\!\!\bigcirc$.
If the system $\mathcal{X}$ is UN with respect to $\rightsquigarrow$ and normalising with respect to $\overset{\star}{\rightsquigarrow}$, then it verifies the Church-Rosser property with respect to $\rightsquigarrow$.

Proof. Let $a, b \in X$ such that $a \overset{\star}{\leftrightsquigarrow} b$. By the hypothesis of normalisation, there exists $n_a, n_b \in \mathrm{NF}(\mathcal{X})$ such that $a \overset{\star}{\rightsquigarrow} n_a$ and $b \overset{\star}{\rightsquigarrow} n_b$. Thus, we get $n_a \overset{\star}{\leftrightsquigarrow} n_b$. But since by hypothesis the system is UN with respect to $\rightsquigarrow$, then $n_a = n_b$. Hence, in summary:



☐

Corollary 2.41

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \;\subseteq\; \rightsquigarrow \;\subseteq\; \overset{\star}{\longrightarrow}\!\!\!\!\bigcirc$. Let $\underset{\mathrm{sub}}{\rightsquigarrow}$ be a relation on X such that $\longrightarrow \;\subseteq\; \underset{\mathrm{sub}}{\rightsquigarrow} \;\subseteq\; \rightsquigarrow$.
Assume that $\rightsquigarrow$ and $\underset{\mathrm{sub}}{\rightsquigarrow}$ generate the same equivalence relation (*i.e.* we have $\overset{\star}{\leftrightsquigarrow} \;\subseteq\; \underset{\mathrm{sub}}{\overset{\star}{\leftrightsquigarrow}}$).
If we have:

(i) (X, τ) is Fréchet,

(ii) the system $\mathcal{X}$ is normalising with respect to $\overset{\star}{\rightsquigarrow}$,

then we have:

$$\text{C-R w.r.t } \rightsquigarrow \quad \Leftrightarrow \quad \text{NF w.r.t } \rightsquigarrow \quad \Leftrightarrow \quad \text{UN w.r.t } \rightsquigarrow \quad \Leftrightarrow \quad \text{UN w.r.t } \underset{\text{sub}}{\rightsquigarrow}.$$

In particular, if the system is normalising with respect to $\underset{\text{sub}}{\overset{\star}{\rightsquigarrow}}$ in addition, then all properties (C-R, NF and UN) with respect to either $\rightsquigarrow$ or $\underset{\text{sub}}{\rightsquigarrow}$ are equivalent.

Corollary 2.42

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
If we have:

(i) (X, τ) is Fréchet,

(ii) the system $\mathcal{X}$ is topologically normalising (TN),

then we have:

$$\text{topo. C-R} \quad \Leftrightarrow \quad \text{TNF} \quad \Leftrightarrow \quad \text{TUN}$$

Furthermore, if the relation $\longrightarrow\!\!\!\odot$ is transitive, then we have: infinitary conf. $\Leftrightarrow$ topo. C-R.

2.3 Rewriting on uniform spaces

Definition 2.43 : Filter

Let X be a set.
A **filter** on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ of subsets of X such that it satisfies the following axioms:

(F-I) Closure under supersets: $\forall A \subseteq X, \quad (\exists B \in \mathcal{F}, B \subseteq A) \;\Rightarrow\; A \in \mathcal{F}$,

(F-II) Closure under finite intersections: $\forall A, B \in \mathcal{F}, \quad A \cap B \in \mathcal{F}$ and $X \in \mathcal{F}$,

(F-III) Does not contain the empty set: $\varnothing \notin \mathcal{F}$.

For instance, if A is a non-empty subset of a topological space X, then the set of all neighbourhoods of A is a filter on X.
Uniform spaces are a generalisation of both metric spaces and topological groups.

Definition 2.44 : Uniform structure

Let X be a set.
A **uniform structure** on X is a collection $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ of binary relations on X (called **entourages**) such that $\mathcal{U}$ is a *filter* on $X \times X$ that satisfies the following axioms:

(U-I) Every entourage is reflexive: $\forall V \in \mathcal{U}, \quad \forall x \in X, \quad xVx$,

(U-II) The inverse of an entourage is an entourage: $\forall V \in \mathcal{U}, \quad V^{-1} \in \mathcal{U}$
where V^{-1} is the relation such that $xV^{-1}y$ if and only if yVx.

(U-III) There always exists a "half-size" entourage: $\forall V \in \mathcal{U}, \quad \exists W \in \mathcal{U}, \quad W \circ W \subseteq V$

where we have $x(W \circ W)z$ if and only if, there exists $y \in X$ with xWy and yWz.

Definition 2.45 : Uniform space

Let (X, τ) be a topological space.
A uniform structure $\mathcal{U}$ on X is said to be **compatible with the topology** τ if:

$$\forall U \subseteq X, \quad U \text{ open in } (X, \tau) \;\Leftrightarrow\; (\forall x \in U, \quad \exists V \in \mathcal{U}, \quad V(x) \subseteq U),$$

where $V(x) := \{y \in X \mid xVy\}$.
If such a structure exists, we say that the topological space (X, τ) is **uniformisable**.
The data of $(X, \tau, \mathcal{U})$ where (X, τ) is a uniformisable space and $\mathcal{U}$ is a uniform structure on X compatible with the topology τ is called a **uniform space**.

Example 2.46 : (Metric spaces are uniformisable)

Let (X, d) be a metric space. Write τ_d the topology on X induced by the metric d.
Consider the family $\mathcal{U} := (V_r)_{r \in \mathbb{R}_{>0}}$ of binary relations on X defined as, for $r \in \mathbb{R}_{>0}$:

$$\forall x, y \in X, \quad xV_ry \overset{\text{def}}{\Leftrightarrow} d(x, y) \leqslant r.$$

Then, $\mathcal{U}$ is a uniform structure on X that is compatible with the topology τ_d.

Example 2.47 : (Topological groups are uniformisable)

Let (G, τ) be a topological group written multiplicatively. Write $1 \in G$ the identity element.
Consider the set $\mathcal{U}$ of binary relations X defined as:

$$\forall V \subseteq G \times G, \quad V \in \mathcal{U} \overset{\text{def}}{\Leftrightarrow} \exists U \in \mathcal{N}(1), \quad \left\{(x, y) \in G \times G \,\middle|\, x \cdot y^{-1} \in U\right\} \subseteq V.$$

Then, $\mathcal{U}$ is a uniform structrue on G that is compatible with the topology τ.

Definition 2.48 : Attractive Normal Forms (ANF)

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{*}{\longrightarrow}$.
The system $\mathcal{X}$ has **globally attractive normal forms (GANF) with respect to** $\rightsquigarrow$ if there exists a uniform structure $\mathcal{U}$ on X compatible with the topology τ such that:

$$\forall n \in \mathrm{NF}(\mathcal{X}), \quad \forall a, b \in X, \quad \forall V \in \mathcal{U}, \quad aVn \;\wedge\; a \rightsquigarrow b \;\Rightarrow\; bVn.$$

The system $\mathcal{X}$ has **locally attractive normal forms (LANF) with respect to** $\rightsquigarrow$ if there exists a uniform structure $\mathcal{U}$ on X compatible with the topology τ such that:

$$\forall n \in \mathrm{NF}(\mathcal{X}), \quad \exists V_n \in \mathcal{U}, \quad \forall a \in V_n(n), \quad \forall b \in X, \quad \forall V \in \mathcal{U}$$
$$aVn \;\wedge\; a \rightsquigarrow b \;\Rightarrow\; bVn$$

It is quite straightforward to see that GANF with respect to relation implies LANF with respect to that same relation.

Remark 2.49

If $\underset{\text{sub}}{\rightsquigarrow}$ is a relation such that $\longrightarrow \subseteq \underset{\text{sub}}{\rightsquigarrow} \subseteq \rightsquigarrow$ and if the system is GANF (resp. LANF) with respect to $\rightsquigarrow$, then it is also GANF (resp. LANF) with respect to $\underset{\text{sub}}{\rightsquigarrow}$.
Also, by simple inductive reasoning, we see that a system is GANF (resp. LANF) with respect to a relation $\rightsquigarrow$ if, and only if, it is GANF (resp. LANF) with respect to the reflexive transitive closure $\overset{\star}{\rightsquigarrow}$.

Proposition 2.50

Let $\mathfrak{X} := (X, \tau_{\text{dis}}^X, \longrightarrow)$ be a topological rewriting system where τ_{dis}^X is the discrete topology. Then, the system $\mathfrak{X}$ has LANF with respect to $\longrightarrow$.

Theorem 2.51

Let $\mathfrak{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\star}{\twoheadrightarrow}$. Let $\underset{\text{sub}}{\rightsquigarrow}$ be a relation on X such that $\longrightarrow \subseteq \underset{\text{sub}}{\rightsquigarrow} \subseteq \rightsquigarrow$. Assume that $\rightsquigarrow$ and $\underset{\text{sub}}{\rightsquigarrow}$ generate the same equivalence relation (*i.e.* we have $\overset{\star}{\leftrightsquigarrow} \subseteq \underset{\text{sub}}{\overset{\star}{\leftrightsquigarrow}}$).
If the following is verified for $\mathfrak{X}$:

(i) (X, τ) is *Hausdorff*,

(ii) the system $\mathfrak{X}$ is normalising with respect to $\underset{\text{sub}}{\overset{\star}{\rightsquigarrow}}$,

(iii) the relation $\twoheadrightarrow$ is transitive (*i.e.* $\overset{\star}{\twoheadrightarrow} \subseteq \twoheadrightarrow$),

(iv) the system $\mathfrak{X}$ has locally attractive normal forms (LANF) with respect to $\rightsquigarrow$,

then, we have the following equivalent properties for $\mathfrak{X}$:

$$\text{finitary confluence w.r.t } \overset{\star}{\rightsquigarrow} \ \Leftrightarrow\ \text{Church-Rosser property w.r.t } \underset{\text{sub}}{\rightsquigarrow}$$

Proof. The right-to-left direction has been shown in Corollary 2.35.

Now, for the converse direction, according to Corollary 2.41, it suffices to show that finitary confluence w.r.t $\overset{\star}{\rightsquigarrow}$ implies UN with respect to $\rightsquigarrow$ under the assumptions (i) — (iv) (because Hausdorff implies Fréchet). Write $\mathfrak{U}$ the uniform structure on X compatible with the topology τ that is implied by the assumption of LANF with respect to $\rightsquigarrow$.

Assume the system $\mathfrak{X}$ is finitary confluent with respect to $\overset{\star}{\rightsquigarrow}$. Show that hypotheses (i) — (iv) imply UN with respect to $\rightsquigarrow$. In order to prove this, use Lemma 2.37 since by hypothesis (i) the space is Fréchet and by hypothesis (ii) the system is normalising with respect to $\underset{\text{sub}}{\overset{\star}{\rightsquigarrow}}$ and thus with respect to $\overset{\star}{\rightsquigarrow}$ as well.

Let $a \in X$ and $n, n' \in \text{NF}(\mathfrak{X})$ such that $n \overset{\star}{\leftsquigarrow} a \overset{\star}{\rightsquigarrow} n'$ and show that $n = n'$. By LANF with respect to $\underset{\text{sub}}{\rightsquigarrow}$, there exist V_n and $V_{n'}$ in $\mathfrak{U}$ such that for all $V \in \mathfrak{U}$ and for all $b, c \in X$ such that nV_nb (resp. $n'V_{n'}b$), bVn (resp. bVn') and $b \underset{\text{sub}}{\rightsquigarrow} c$, we have cVn (resp. cVn').

Let U and U' be neighbourhoods of n and n' respectively. By compatibility with the topology, there exists V and V' in $\mathfrak{U}$ such that $V(n) \subseteq U$ and $V'(n') \in U'$. Now, denote $W := V_n \cap V$ and $W' := V_{n'} \cap V'$. Note how $W(n)$ and $W'(n')$ are neighbourhoods of n and n' respectively that verify $W(n) \subseteq U$ and $W'(n') \subseteq U'$.

Now, since $\underset{\text{sub}}{\rightsquigarrow} \subseteq \twoheadrightarrow$ and $\twoheadrightarrow$ is transitive by hypothesis (iii), we get $n \twoheadleftarrow a \twoheadrightarrow n'$. It follows that there exists $(b, b') \in W(n) \times W'(n')$ such that $b \overset{\star}{\longleftarrow} a \overset{\star}{\longrightarrow} b'$. By assumption, there exists $c \in X$ such that $b \overset{\star}{\rightsquigarrow} c \overset{\star}{\leftsquigarrow} b'$. But, since bWn and $b'W'n'$ by construction, then we get, by hypothesis (iv), cWn and $cW'n'$. That is to say, we have $c \in W(n) \cap W'(n')$. But recall that

$W(n) \cap W'(n') \subseteq U \cap U'$. In other words, we proved that, for any U and U' neighbourhoods of n and n' respectively, $U \cap U' \neq \varnothing$. Hence, by the Hausdorff property, it follows that $n = n'$. $\square$

According to Corollary 2.35 and Corollary 2.41, we obtain the following equivalences between notions:

Corollary 2.52

With the same hypotheses (i) — (iv) and notations, then all the following statements are equivalent in the system $\mathcal{X}$:

- finitary confluence with respect to $\overset{\star}{\rightsquigarrow}$ (or $\underset{\text{sub}}{\overset{\star}{\rightsquigarrow}}$),
- Church-Rosser property with respect to $\rightsquigarrow$ (or $\underset{\text{sub}}{\rightsquigarrow}$),
- normal form (NF) property with respect to $\rightsquigarrow$ (or $\underset{\text{sub}}{\rightsquigarrow}$),
- unique normal form (UN) property with respect to $\rightsquigarrow$ (or $\underset{\text{sub}}{\rightsquigarrow}$).

In particular, if we take $\rightsquigarrow$ and $\underset{\text{sub}}{\rightsquigarrow}$ both as equal to $\multimap$, then we obtain the following corollary:

Corollary 2.53

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
If the following is verified for $\mathcal{X}$:

(i) (X, τ) is *Hausdorff*,

(ii) the system $\mathcal{X}$ is topologically normalising (TN),

(iii) the relation $\multimap$ is transitive (*i.e.* $\overset{\star}{\multimap} \subseteq \multimap$),

(iv) the system $\mathcal{X}$ has locally attractive normal forms (LANF) with respect to $\multimap$,

then, we have the following equivalent properties for $\mathcal{X}$:

topological confluence $\Leftrightarrow$ topological Church-Rosser property $\Leftrightarrow$ infinitary confluence.

2.4 Limit rewriting induces rewriting chains

Definition 2.54 : Topological rewriting with chains

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
We define the **topological rewriting with chains relation**, denoted by $\rightarrowtail$, as, for $a, b \in X$:

$$a \rightarrowtail b \overset{\text{def}}{\Leftrightarrow} \exists (c_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}, \begin{cases} c_0 = a, \\ c_n \overset{=}{\longrightarrow} c_{n+1} & \forall n \in \mathbb{N}, \\ \lim_{n\to\infty} c_n = b. \end{cases}$$

Write $\overset{\star}{\leftarrowtail\!\rightarrowtail}$ the equivalence relation generated by $\rightarrowtail$.

By Proposition 2.10, it is easy to see that $\rightarrowtail \subseteq \multimap$. It follows that $\overset{\star}{\leftarrowtail\!\rightarrowtail} \subseteq \overset{\star}{\leftarrow\!\multimap}$ by properties of generated equivalence relations. It also entails that the converse of that latter inclusion is verified if and only if $\multimap \subseteq \overset{\star}{\leftarrowtail\!\rightarrowtail}$.

However, consider the following example that shows that the converses inclusions are not true in general:

Example 2.55

Consider $X := [0\,;1]$ endowed with the euclidean topology τ. Consider the topological rewriting system $\mathcal{X} := (X, \tau, \longrightarrow)$ where $\longrightarrow$ is the relation:

$$1 \longrightarrow \frac{1}{2^n} \qquad \forall n \geqslant 1.$$

Then we have $1 \mathrel{-\!\!\circledast} 0$ because for any $\varepsilon > 0$, there exists $n_\varepsilon \geqslant 1$, such that $\varepsilon > \frac{1}{2^{n_\varepsilon}}$ and we conclude since we have $1 \longrightarrow \frac{1}{2^{n_\varepsilon}}$.
However, we do not have $1 \mathrel{\rightarrowtail\!\!\circledast} 0$ because every $\frac{1}{2^n}$ for $n \geqslant 1$ is a normal form.
Moreover, we have $\mathrel{\circledast\!\!-\!\overset{\star}{\diamond}\!-\!\!\circledast} = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^m}\right) \,\middle|\, n, m \in \mathbb{N} \right\}$ and hence $(1, 0) \notin \mathrel{\circledast\!\!-\!\overset{\star}{\diamond}\!-\!\!\circledast}$ thus $\mathrel{\circledast\!\overset{\star}{\text{---}}\!\circledast} \not\subseteq \mathrel{\circledast\!\!-\!\overset{\star}{\diamond}\!-\!\!\circledast}$.

This motivates the following definition:

Definition 2.56 : Chains exist

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
We say that **chains exist in the system** $\mathcal{X}$ if:

$$\forall a, b \in X, \quad a \mathrel{-\!\!\circledast} b \;\Rightarrow\; a \mathrel{\rightarrowtail\!\!\circledast} b.$$

Consider the following notion:

Definition 2.57 : Rewriting stability

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \mathrel{-\!\!\circledast}$.
We say that the relation $\rightsquigarrow$ has **rewriting stability** if:

$$\forall a, b \in X, \quad a \rightsquigarrow b \;\Rightarrow\; \forall U \in \mathcal{N}(b), \quad \exists c \in U, \quad a \xrightarrow{\star} c \rightsquigarrow b.$$

Example 2.58

The topological rewriting with chains relation has rewriting stability. Indeed, let $a, b \in X$ and U be a neighbourhood of b. By definition, if $a \mathrel{\rightarrowtail\!\!\circledast} b$ there exists a sequence $(c_n)_{n\in\mathbb{N}}$ such that $c_0 = a$, $c_n \xrightarrow{=} c_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = b$. By that latter limit condition, we deduce that there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, we have $c_n \in U$. But by the rest of the conditions, it follows, on one hand, that $a = c_0 \xrightarrow{=} c_1 \xrightarrow{=} c_2 \xrightarrow{=} \cdots \xrightarrow{=} c_N$, hence, $a \xrightarrow{\star} c_N$ and, on the other hand, $c_N \mathrel{\rightarrowtail\!\!\circledast} b$ since the sequence $(c_{N+n})_{n\in\mathbb{N}}$ satisfies the definition.

In fact, when the underlying topological space is first-countable, any other relation that has rewriting stability is a subrelation of $\mathrel{\rightarrowtail\!\!\circledast}$. In particular, that latter relation is the biggest relation that has rewriting stability, as shown in the following proposition:

Proposition 2.59

Let $\mathcal{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \mathrel{-\!\!\circledast}$.

Assume that the space (X, τ) is *first-countable*.
Then, the relation $\rightsquigarrow$ has rewriting stability if, and only if, $\rightsquigarrow \subseteq \rightarrowtail\!\!\!\oplus$.

Proof. As shown in the previous example, the relation $\rightarrowtail\!\!\!\oplus$ has rewriting stability. Therefore, it is clear that any subrelation has that property as well, which gives us the right-to-left direction.

For the left-to-right direction, consider a relation $\rightsquigarrow$ that has rewriting stability.

Let $a, b \in X$ such that $a \rightsquigarrow b$. Since, by hypothesis, (X, τ) is first-countable there exists $\left(\mathcal{B}_n^b\right)_{n\in\mathbb{N}}$ a countable neighbourhood basis of b that is totally ordered for the $\supseteq$ relation. Let us now construct a rewriting chain $(c_k)_{k\in\mathbb{N}}$ from a to b to show that indeed $a \rightarrowtail\!\!\!\oplus b$.

Base step is to set $c_0 := a$. Now, the inductive step consists in assuming that the sequence $(c_k)_{k\in\mathbb{N}}$ has been constructed up to and including the step $K \in \mathbb{N}$ in such a way that for all $k \in [\![0\,..\,K-1]\!]$ we have $c_k \xrightarrow{=} c_{k+1}$ and $c_K \rightsquigarrow b$. Consider the following set:

$$M_K := \left\{ n \in \mathbb{N} \;\middle|\; c_K \notin \mathcal{B}_n^b \right\}.$$

If M_K is empty, this means that $c_K \in \bigcap_{n\in\mathbb{N}} \mathcal{B}_n^b$. Therefore, we can set $c_k := c_K$ for all $k \geqslant K$ and we get the desired result.

Otherwise, M_K is a non-empty set of natural numbers. Therefore, it must admit a minimum; let us write it $N := \min M_K$. Then, since $c_K \rightsquigarrow b$ by induction hypothesis, we deduce using the assumption that we have rewriting stability that there exists $c \in \mathcal{B}_N^b$ such that $c_K \xrightarrow{\star} c \rightsquigarrow b$ with $c_K \neq c$ (by defintion of N). If $\ell \geqslant 1$ is the length of the chain betweeen c_K and c (*i.e.* we have $c_K = d_0 \longrightarrow d_1 \longrightarrow \cdots \longrightarrow d_\ell = c$), then define the ℓ next terms of the $(c_k)_{k\in\mathbb{N}}$ sequence by $c_{K+i} = d_i$ for all $i \in [\![1\,..\,\ell]\!]$. It is clear that induction hypothesis is verified for the step $K + \ell$. Thus we can continue.

Repeating this process *ad infinitum* yields a sequence $(c_k)_{k\in\mathbb{N}}$ that is:

- either stationary in a point that is not separated (in the topological sense) from b; hence the sequence converges to b,
- or not stationary but it still converges to b because the c_K's are chosen to be always in a neighbourhood $\mathcal{B}_N^b$ that is strictly contained in the neighbourhood of the previous c_K. But since $\left(\mathcal{B}_n^b\right)_{n\in\mathbb{N}}$ is a neighbourhood basis of b, it follows that the sequence $(c_k)_{k\in\mathbb{N}}$ converges to b.

The constructed sequence $(c_k)_{k\in\mathbb{N}}$ is thus such that $c_0 = a$, $c_k \xrightarrow{=} c_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} c_k = b$, *i.e.* $a \rightarrowtail\!\!\!\oplus b$. ☐

Corollary 2.60

Let $\mathfrak{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system.
Assume that (X, τ) is *first-countable*.
Then, chains exists in $\mathfrak{X}$ if, and only if, $\longrightarrow\!\!\!\!\oplus$ has rewriting stability.

Proof. The left-to-right direction does not require the first-countability hypothesis because it implies that $\longrightarrow\!\!\!\!\oplus$ is equal to $\rightarrowtail\!\!\!\oplus$ which always has rewriting stability.

The other direction is direct consequence of the previous proposition. ☐

We can see how we utilise the fact that the repetition of the induction step yields a well-defined rewriting chain because, at each step, we get closer with respect to the neighbourhood basis. Let us introduce the notion of "proximity maps" in first-countable spaces to explicit a characterisation of rewriting stability that does not depend on explicitly checking all neighbourhood of the limit point.

Definition 2.61 : Proximity map

Let (X, τ) be a *first-countable* topological space.
We say that we define **proximity maps on** X when we fix for each $x \in X$, a countable neighbourhood basis $(\mathcal{B}_n^x)_{n\in\mathbb{N}}$ of x that is totally ordered for $\supseteq$ and define the **proximity map for** x **with respect to the choosen basis** $(\mathcal{B}_n^x)_{n\in\mathbb{N}}$ as:

$$\begin{aligned} P_x : X &\to \mathbb{N} \cup \{\infty\} \\ y &\mapsto P_x(y) := \begin{cases} \infty & \text{if } y \in \bigcap_{n\in\mathbb{N}} \mathcal{B}_n^x, \\ \min\{n \in \mathbb{N} \mid y \notin \mathcal{B}_n^x\} & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem 2.62

Let $\mathfrak{X} := (X, \tau, \longrightarrow)$ be a topological rewriting system and $\rightsquigarrow$ be a binary relation on X such that $\longrightarrow \subseteq \rightsquigarrow \subseteq \overset{\circledast}{\longrightarrow}$.
Assume that (X, τ) is *first-countable* and define proximity maps P_x for $x \in X$.
Then, $\rightsquigarrow$ has rewriting stability if and only if for all $a, b \in X$ such that $a \rightsquigarrow b$, there exists $c \in X$ that satisfy the following conditions:

(i) $a \xrightarrow{\star} c \rightsquigarrow b$,

(ii) $P_b(c) > P_b(a)$.

2.5 Topological Newman's lemma

Todo

3 Rewriting on commutative formal power series in several variables

3.1 Construction of commutative formal power series in several variables

Todo: Cauchy completion on the multivariate polynomials in commuting variables $\{x_1, \cdots, x_n\}$ with respect to the metric inducing the $(x_1, \cdots, x_n)$-adic topology. Zariski ring (*i.e.* ideals are topologically closed for the adic topology).

Write $\mathbb{K}[[x_1, \cdots, x_n]]$ the topological algebra of commutative formal power series in n variables with respect to the topology τ_δ induced by the metric δ:

$$\forall f, g \in \mathbb{K}[[x_1, \cdots, x_n]], \quad \delta(f, g) := \frac{1}{2^{\mathrm{val}(f-g)}},$$

where $\mathrm{val}(h)$ is the least degree of monomials appearing in the support of a non-zero formal power series h. We set by convention $\mathrm{val}(0) = \infty$ and $\frac{1}{2^\infty} = 0$.

3.2 Reduction on commutative formal power series

Write $[x_1, \cdots, x_n]$ the commutative monoid generated by $\{x_1, \cdots, x_n\}$ written multiplicatively and with identity element 1. We call **monomials** the elements of $[x_1, \cdots, x_n]$.

For a formal power series $f \in \mathbb{K}[[x_1, \cdots, x_n]]$ and a monomial $m \in [x_1, \cdots, x_n]$, we write $\langle f|m\rangle \in \mathbb{K}$ the coefficient of m in f.

A **monomial order** is a total order $<$ on $[x_1, \cdots, x_n]$ such that it is compatible with monomial multiplication:

$$\forall m, m_1, m_2 \in [x_1, \cdots, x_n], \quad m_1 < m_2 \;\Rightarrow\; m \cdot m_1 < m \cdot m_2.$$

The opposite order $<^{\mathrm{op}}$ of a monomial order $<$ is a monomial order.

A monomial order is said to be **compatible with the degree** if the degree function on $[x_1, \cdots, x_n]$ is increasing:

$$\forall m_1, m_2 \in [x_1, \cdots, x_n], \quad m_1 \leqslant m_2 \;\Rightarrow\; \deg(m_1) \leqslant \deg(m_2).$$

We say that a monomial order is **admissible** if 1 is minimal:

$$\forall m \in [x_1, \cdots, x_n] \setminus \{1\}, \quad 1 < m.$$

This is equivalent to saying that $<$ is a well-order.

For a monomial order $<$, we denote by $\mathrm{LM}_<(f)$ the maximal element for $<$ in the support of any non-zero formal power series f (provided it exists). If it exists, we write $\mathrm{LC}_<(f)$ its coefficient in f and $\mathrm{LT}_<(f) := \mathrm{LC}_<(f)\,\mathrm{LM}_<(f)$ as well as $\mathrm{r}_<(f) := \mathrm{LT}_<(f) - f$.

If $<$ is an admissible monomial order and f is an infinite formal power series, then $\mathrm{LM}_<(f)$ is not defined; however, $\mathrm{LM}_{<^{\mathrm{op}}}(f)$ is.

Also, note how if $<$ is an admissible monomial order compatible with the degree, then we have for any non-zero formal power series f:

$$\mathrm{val}(f) = \deg\left(\mathrm{LM}_{<^{\mathrm{op}}}(f)\right).$$

Let R be non-empty set of non-zero formal power series in $\mathbb{K}[[x_1, \cdots, x_n]]$ and let $<$ be an admissible monomial order.

We define the following relation $\underset{R}{\rightarrow}$ on $\mathbb{K}[[x_1, \cdots, x_n]]$, called **multivariate series reduction**:

$$\lambda(m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)) + S \quad \underset{R}{\rightarrow} \quad \frac{\lambda}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times \mathrm{r}_{<^{\mathrm{op}}}(s) + S$$

where:

- $\lambda \in \mathbb{K} \setminus \{0\}$,
- $m \in [x_1, \cdots, x_n]$,
- $s \in R$,
- $S \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)$ is not in its support.

We say that we **rewrite the monomial** $M := m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)$ **using the rule** s **in the formal power series** $f := \lambda(m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)) + S$.

Proposition 3.1

The relation $\underset{R}{\rightarrow}$ is *anti-reflexive.*

Consider the topological rewriting system $\mathcal{X}_R := (\mathbb{K}[[x_1, \cdots, x_n]], \tau_\delta, \underset{R}{\rightarrow})$.

Denote by:

- $\overset{\star}{\underset{R}{\rightarrow}}$ the reflexive transitive closure of $\underset{R}{\rightarrow}$, ($\overset{\star}{\underset{R}{\leftrightarrow}}$ the generated equivalence relation),
- $\underset{R}{\rightarrow\!\!\!\!\!\!\circ}$ the topological rewriting relation associated with $\mathcal{X}_R$, ($\overset{\star}{\underset{R}{\circ\!\!\leftrightarrow\!\!\circ}}$ the generated equivalence relation),

- $\underset{R}{\rightarrowtail\!\!\circlearrowright}$ the topological rewriting with chains relation associated with $\mathcal{X}_R$, ($\underset{R}{\overset{\star}{\circlearrowleft\!\!\leftrightarrow\!\!\circlearrowright}}$ the generated equivalence relation),

- $\underset{R}{\rightarrowtail\!\!\circleddash}$ the relation defined as follows:

$$f \underset{R}{\rightarrowtail\!\!\circleddash} g \overset{\text{def}}{\Leftrightarrow} \exists S := \{s_1, \cdots, s_r\} \subseteq R, \quad f \underset{S}{\rightarrowtail\!\!\circlearrowright} g.$$

($\underset{R}{\overset{\star}{\circleddash\!\!\leftrightarrow\!\!\circleddash}}$ the generated equivalence relation),

We have: $\underset{R}{\rightarrow} \subseteq \overset{\star}{\underset{R}{\rightarrow}} \subseteq \underset{R}{\rightarrowtail\!\!\circleddash} \subseteq \underset{R}{\rightarrowtail\!\!\circlearrowright} \subseteq \underset{R}{\rightarrow\!\!\circlearrowright}$.

3.3 Deciding the ideal membership problem

Lemma 3.2

If the order $<$ is compatible with the degree, then for any $h \in \mathbb{K}[[x_1, \cdots, x_n]]$ and any $s \in R$, we have $h \times s \underset{R}{\rightarrowtail\!\!\circleddash} 0$.

Proof. Write $\operatorname{supp}(h)$ as the sequence $(m_k)_k$ strictly increasing (possible because the order is of type ω). If we write $u := \operatorname{card}(\operatorname{supp}(h))$, then:

$$h = \sum_{k=0}^{u} \langle h|m_k \rangle \, m_k$$

and thus:

$$h \times s = \sum_{k=0}^{u} \langle h|m_k \rangle \, m_k \times s.$$

If $u < \infty$, then it clear that $h \times s \overset{\star}{\underset{\{s\}}{\rightarrow}} 0$ by rewriting successively the m_k with k increasing.

If $u = \infty$, then we define the infinite sequence $(r_k)_{k \in \mathbb{N}}$ as $r_0 := h \times s$ and for any $k \in \mathbb{N}$:

$$r_{k+1} := r_k - \frac{\mathrm{LC}_{<^{\mathrm{op}}}(r_k)}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m_k \times s.$$

Then, we see that $h \times s = r_0 \underset{\{s\}}{\rightarrow} r_1 \underset{\{s\}}{\rightarrow} r_2 \underset{\{s\}}{\rightarrow} \cdots$.

Since the order is compatible with the degree, it follows that the sequence $(r_k)_{k \in \mathbb{N}}$ is Cauchy and therefore has a limit since $\mathbb{K}[[x_1, \cdots, x_n]]$ is complete. Moreover, since m_k is strictly increasing and $\mathrm{LM}_{<^{\mathrm{op}}}(r_k) = m_k \times \mathrm{LM}_{<^{\mathrm{op}}}(s)$, we get that:

$$\lim_{k \to \infty} r_k = 0.$$

Hence, $h \times s \underset{\{s\}}{\rightarrowtail\!\!\circlearrowright} 0$, *i.e.* $h \times s \underset{R}{\rightarrowtail\!\!\circleddash} 0$. □

Lemma 3.3 : (Translation lemma)

If the order $<$ is compatible with the degree, then for all $f, g, h \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f - g \underset{R}{\rightarrowtail\!\!\circleddash} h$ then there exist $f', g' \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $h = f' - g'$, $f \underset{R}{\rightarrowtail\!\!\circleddash} f'$ and $g \underset{R}{\rightarrowtail\!\!\circleddash} g'$.

Proof. Since $f - g \underset{R}{\rightarrowtail\!\!\circleddash} h$, there exists $S := \{s_1, \cdots, s_r\} \subseteq R$ and a sequence $(h_k)_{k \in \mathbb{N}}$ of formal power series such that $h_0 = f - g$, $h_k \overset{=}{\underset{S}{\rightarrow}} h_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k \in \infty} h_k = h$.

Let us contruct two new sequences $(f_k)_{k \in \mathbb{N}}$ and $(g_k)_{k \in \mathbb{N}}$. Set $f_0 := f$ and $g_0 := g$.

Let $k \in \mathbb{N}$.

If $h_k = h_{k+1}$, then set $f_{k+1} := f_k$ and $g_{k+1} := g_k$.

Otherwise, there must exist $m \in [x_1, \cdots, x_n]$ and $s \in S$ such that $h_{k+1} = h_k - \frac{\langle h_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s$. Then, set:

$$f_{k+1} := f_k - \frac{\langle f_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s, \qquad g_{k+1} := g_k - \frac{\langle g_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s.$$

Hence, we get $f_k \xrightarrow[S]{=} f_{k+1}$ and $g_k \xrightarrow[S]{=} g_{k+1}$. Therefore, the sequences $(f_k)_{k \in \mathbb{N}}$ and $(g_k)_{k \in \mathbb{N}}$ are Cauchy by compatibility with the degree. They thus admit limits; write them f' and g' respectively. In particular, we have $f \underset{S}{\rightarrowtail\!\ominus} f'$ and $g \underset{S}{\rightarrowtail\!\ominus}$, hence: $f \underset{R}{\rightarrowtail\!\ominus} f'$ and $g \underset{R}{\rightarrowtail\!\ominus} g'$.

Let us show by induction that for all $k \in \mathbb{N}$, we have $h_{k+1} = f_{k+1} - g_{k+1}$. The base step is verified. Let $k \in \mathbb{N}$ such that $h_k = f_k - g_k$. Then:

$$\begin{aligned}
h_{k+1} &= h_k - \frac{\langle h_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s \\
&= (f_k - g_k) - \frac{\langle f_k - g_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s \\
&= \left(f_k - \frac{\langle f_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s \right) - \left(g_k - \frac{\langle g_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s \right) \\
h_{k+1} &= f_{k+1} - g_{k+1}
\end{aligned}$$

Therefore, we get:

$$h = \lim_{k \in \mathbb{N}} h_k = \lim_{k \in \mathbb{N}} (f_k - g_k) = \lim_{k \in \mathbb{N}} f_k - \lim_{k \in \mathbb{N}} g_k = f' - g'.$$

Hence, the desired result. □

A direct consequence of this Translation lemma is the case where $h = 0$:

Corollary 3.4 : Translation corollary

If the order $<$ is compatible with the degree, then for all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f - g \underset{R}{\rightarrowtail\!\ominus} 0$, there exists $h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \underset{R}{\rightarrowtail\!\ominus} h \underset{R}{\ominus\!\leftarrowtail} g$.

Denote by $\equiv_{I(R)}$ the congruence relation modulo the ideal in $\mathbb{K}[[x_1, \cdots, x_n]]$ generated by R.

Theorem 3.5

If the order $<$ is compatible with the degree, then for all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \equiv g \pmod{I(R)}$, we have $f \overset{\star}{\underset{R}{\ominus\!\!-\!\!\diamond\!\!-\!\!\ominus}} g$. In other words, we have $\equiv_{I(R)} \subseteq \overset{\star}{\underset{R}{\ominus\!\!-\!\!\diamond\!\!-\!\!\ominus}}$.

Proof. Let us show by induction on r that for all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, for all $\{s_1, \cdots, s_r\} \subseteq R$ and for all $(q_1, \cdots, q_r) \in \mathbb{K}[[x_1, \cdots, x_n]]^r$ such that $g - f = \sum_{i=1}^{r} q_i \times s_i$, we have $f \overset{\star}{\underset{R}{\ominus\!\!-\!\!\diamond\!\!-\!\!\ominus}} g$.

Base step: if $r = 0$, then we always have $f = g$ and there is nothing to prove since $\overset{\star}{\underset{R}{\ominus\!\!-\!\!\diamond\!\!-\!\!\ominus}}$ is reflexive being an equivalence relation.

Induction step: let $r \in \mathbb{N}$ and assume the induction hypothesis for r. Let $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$ and $\{s_1, \cdots, s_{r+1}\} \subseteq R$ and $(q_1, \cdots, q_{r+1}) \in \mathbb{K}[[x_1, \cdots, x_n]]^{r+1}$ such that $g - f = q_{r+1} \times s_{r+1} + \sum_{i=1}^{r} q_i \times s_i$.

By Lemma 3.2, we have $q_{r+1} \times s_{r+1} \underset{R}{\rightarrowtail\!\ominus} 0$.

Thus, we can apply the Corollary 3.4 of translation on $f - f + q_{r+1} \times s_{r+1}$ and we deduce there exists $h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that:

$$f + q_{r+1} \times s_{r+1} \underset{R}{\rightarrowtail\!\ominus} h \underset{R}{\ominus\!\leftarrowtail} f.$$

In particular, we have $f + q_{r+1} \times s_{r+1} \overset{\star}{\underset{R}{\ominus\!\!-\!\!\ominus}} f$.

But, by definition we have:

$$f + q_{r+1} \times s_{r+1} = g - \sum_{i=1}^{r} q_i \times s_i.$$

It follows, by induction hypothesis applied for $f := f + q_{r+1} \times s_{r+1}$ and $g := g$, that $f + q_{r+1} \times s_{r+1} \overset{\star}{\underset{R}{\ominus\!\!-\!\!\ominus}} g$.

By transitivity, we finally conclude that $f \overset{\star}{\underset{R}{\ominus\!\!-\!\!\ominus}} g$.

Now let $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$. Assume that $f \equiv g \pmod{I(R)}$. Hence, $g - f \in I(R)$. Hence, there exist $\{s_1, \cdots, s_r\} \subseteq R$ and $(q_1, \cdots, q_r) \in \mathbb{K}[[x_1, \cdots, x_n]]^r$ such that $g - f = \sum_{i=1}^{r} q_i \times s_i$. And therefore, by the previous discussion, since r is finite, we have $f \overset{\star}{\underset{R}{\ominus\!\!-\!\!\ominus}} g$. $\square$

The converse holds as we will see after the following lemma.

Lemma 3.6

For all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f \underset{R}{\longrightarrow\!\!\!\!\!\oplus} g$, then $f - g \in I(R)$, the ideal generated by R.

Proof. First, if $f = g$, then there is nothing to prove. Second, if $f \underset{R}{\rightarrow} g$, then we have:

$$f = \lambda(m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)) + S, \qquad g = \frac{\lambda}{\mathrm{LC}_{<^{\mathrm{op}}}(s)}(m \times \mathrm{r}_{<^{\mathrm{op}}}(s)) + S,$$

for $\lambda \in \mathbb{K} \setminus \{0\}$, $m \in [x_1, \cdots, x_n]$, $s \in R$ and $S \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \notin \mathrm{supp}(S)$.

Cancellations ensue in the computation of $f - g$, and we obtain:

$$f - g = \frac{\lambda}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s.$$

But $s \in R \subseteq I(R)$ and $I(R)$ is an ideal, therefore $f - g \in I(R)$. Third, if $f \overset{\star}{\underset{R}{\rightarrow}} g$ and $f \neq g$, then by induction on the length $k \geqslant 1$ of the rewriting sequence $f = f_0 \longrightarrow f_1 \longrightarrow \cdots \longrightarrow f_k = g$, we have $f - g \in I(R)$. Finally, if we have $f \underset{R}{\longrightarrow\!\!\!\!\!\oplus} g$, then for every integer $k \in \mathbb{N}$, there exists $f_k \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \overset{\star}{\underset{R}{\rightarrow}} f_k$ and:

$$\delta(f_k, g) < \frac{1}{2^k}.$$

The sequence $(f_k)_{k \in \mathbb{N}}$ thus converges to g. From the third case treated in the current proof, for every $k \in \mathbb{N}$, we have $f - f_k \in I(R)$, so that $f - g = \lim_{k \to \infty}(f - f_k)$ belongs to the topological closure $\overline{I(R)}$ of $I(R)$. Now, since **ideals of commutative formal power series are topologically closed**, we have $f - g \in I(R)$. $\square$

Theorem 3.7

For all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$, if $f \overset{\star}{\underset{R}{\oplus\!\!-\!\!\oplus}} g$, then $f \equiv g \pmod{I(R)}$. In other words, we have: $\overset{\star}{\underset{R}{\oplus\!\!-\!\!\oplus}} \subseteq \equiv_{I(R)}$.

Proof. Let $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \overset{\star}{\underset{R}{\oplus\!\!-\!\!\oplus}} g$. Decompose that into the sequence:

$$f = h_0 \underset{R}{\oplus\!\!-\!\!\oplus} h_1 \underset{R}{\oplus\!\!-\!\!\oplus} \cdots \underset{R}{\oplus\!\!-\!\!\oplus} h_\ell = g.$$

By induction on ℓ: if $\ell = 0$, then $f = g$ and thus $g - f = 0 \in I(R)$. Suppose that $f \overset{\ell}{\underset{R}{\leftrightarrow}} h \underset{R}{\leftrightarrow} g$ for $\ell \in \mathbb{N}$ and $h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $h - f \in I(R)$.

Since $h \underset{R}{\leftrightarrow} g$, we have $h \underset{R}{\multimap} g$ or $h \underset{R}{\multimapinv} g$. By Lemma 3.6, we get $h - g \in I(R)$ or $g - h \in I(R)$. Notice how, since $I(R)$ is an ideal, those two latter conditions are actually equivalent. Assume one of them to be true, say $g - h \in I(R)$.

Since $I(R)$ is an ideal and by induction hypothesis, we obtain $I(R) \ni (g - h) + (h - f) = g - f$, hence $f \equiv g \pmod{I(R)}$. □

Corollary 3.8

If the order $<$ is compatible with the degree, then the equivalence relations $\overset{\star}{\underset{R}{\leftrightarrow}}$, $\overset{\star}{\underset{R}{\updownarrow}}$ and $\overset{\star}{\underset{R}{\updownarrow}}$ generated by $\underset{R}{\multimap}$, $\underset{R}{\rightarrowtail}$ and $\underset{R}{\rightarrowtail}$ respectively are actually the same relation: they are all equal to the congruence relation $\equiv_{I(R)}$ modulo the ideal generated by R.

Proof. By Theorem 3.5, Theorem 3.7 and the trivial inclusions of the relations we obtain that:

$$\equiv_{I(R)} \subseteq \overset{\star}{\underset{R}{\updownarrow}} \subseteq \overset{\star}{\underset{R}{\updownarrow}} \subseteq \overset{\star}{\underset{R}{\leftrightarrow}} \subseteq \equiv_{I(R)}$$

Hence, the equality between all of them. □

3.4 Properties of the rewriting relation commutative formal power series

Proposition 3.9

If the order $<$ is *compatible with the degree*, then the relation $\multimap$ of the system $\mathcal{X}_R$ is *transitive*.

Proof. First, let us prove that for all $f, g, h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \underset{R}{\multimap} g \xrightarrow[R]{} h$ we have $f \underset{R}{\multimap} h$.

Let $f, g, h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \underset{R}{\multimap} g$ and $g \xrightarrow[R]{} h$. By definition, that latter relation implies the existence of $m \in [x_1, \cdots, x_n]$ and $s \in R$ such that:

$$h = g - \frac{\langle g | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s.$$

Note how $\mathrm{LM}_{<^{\mathrm{op}}}(g - h) = m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)$.

Let U be an open neighbourhood of h.

If $g \in U$, then U is also a neighbourhood of g and therefore, since $f \underset{R}{\multimap} g$, we obtain $f \underset{R}{\multimap} h$.

Assume thus that $g \notin U$. Therefore, there exists $N_U \in \mathbb{N}$ such that $\overline{B}\left(h, \frac{1}{2^{N_U}}\right) \subsetneq U$ and $\deg\left(\mathrm{LM}_{<^{\mathrm{op}}}(g - h)\right) < N_U$, *i.e.* $\delta(g, h) > \frac{1}{2^{N_U}}$ because the order is compatible with the degree.

Since we have $f \underset{R}{\multimap} g$, there exists a sequence $(f_k)_{k \in \mathbb{N}}$ of formal power series in $\mathbb{K}[[x_1, \cdots, x_n]]$ such that $f_0 = f$ and for all $k \in \mathbb{N}$ $f \overset{\star}{\underset{R}{\to}} f_k$ as well as $\lim_{k \to \infty} f_k = g$.

By that latter limit property, we can assert the existence of $K_{N_U} \in \mathbb{N}$ such that for all $k \geqslant K_{N_U}$ we have $\delta(f_k, g) \leqslant \frac{1}{2^{N_U}}$. But, since also $N_U > \deg\left(\mathrm{LM}_{<^{\mathrm{op}}}(g - h)\right) = \deg\left(m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\right)$, we get $\langle g | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \rangle = \left\langle f_{K_{N_U}} \middle| m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \right\rangle \neq 0$ (non-zero because we rewrite g into h using m and s).

Write:

$$f'_{K_{N_U}} = f_{K_{N_U}} - \frac{\left\langle f_{K_{N_U}} \middle| m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \right\rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s.$$

This is the direct successor of $f_{K_{N_U}}$ that we obtain by rewriting $m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)$ using the rule s.

Then, we compute $h - f'_{K_{N_U}} = g - f_{K_{N_U}}$.

Thus, it follows that $\delta\left(h, f'_{K_{N_U}}\right) = \delta\left(g, f_{K_{N_U}}\right) \leqslant \frac{1}{2^{N_U}}$.

Hence, we have exhibited $f'_{K_{N_U}} \in \overline{B}\left(h, \frac{1}{2^{N_U}}\right) \subset U$ such that $f \xrightarrow[R]{\star} f_{K_{N_U}} \xrightarrow[R]{} f'_{K_{N_U}}$, which conludes the first part of the proof.

By simple induction, it follows that for all f, g, h such that $f \xrightarrow[R]{\circ} g \xrightarrow[R]{\star} h$ we have $f \xrightarrow[R]{\circ} h$.

Now, let $f, g, h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \xrightarrow[R]{\circ} g \xrightarrow[R]{\circ} h$. Then there exists a sequence $(g_k)_{k\in\mathbb{N}}$ such that $g_0 = g$ and for all $k \in \mathbb{N}$ $g \xrightarrow[R]{\star} g_k$ as well as $\lim_{k\to\infty} g_k = h$.

Let $\varepsilon \in \mathbb{R}_{>0}$. Since $\lim_{k\to\infty} g_k = h$, there exists $K_\varepsilon \in \mathbb{N}$ such that $\delta(g_{K_\varepsilon}, h) < \frac{\varepsilon}{2}$. Thus we have $f \xrightarrow[R]{\circ} g \xrightarrow[R]{\star} g_{K_\varepsilon}$. But by the previous part of the proof, it follows that $f \xrightarrow[R]{\circ} g_{K_\varepsilon}$. Now, it follows that there exists a sequence $(f_k)_{k\in\mathbb{N}}$ such that $f_0 = f$ and for all $k \in \mathbb{N}$ $f \xrightarrow[R]{\star} f_k$ as well as $\lim_{k\to\infty} f_k = g_{K_\varepsilon}$. Therefore, there exists $K'_\varepsilon \in \mathbb{N}$ such that $\delta(f_{K'_\varepsilon}, g_{K_\varepsilon}) < \frac{\varepsilon}{2}$. In conclusion, we have on one hand $\delta(f_{K'_\varepsilon}, h) \leqslant \delta(f_{K'_\varepsilon}, g_{K_\varepsilon}) + \delta(g_{K_\varepsilon}, h) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and, on the other hand, $f \xrightarrow[R]{\star} f_{K'_\varepsilon}$. Hence $f \xrightarrow[R]{\circ} h$. ☐

Proposition 3.10

If the order $<$ is *compatible with the degree*, then the system $\mathfrak{X}_R$ is normalising with respect to $\underset{R}{\overset{\star\,\circ}{\rightarrowtail}}$.

Proof. By Dickson's lemma, there exists $S := \{s_1, \cdots, s_r\} \subseteq R$ such that $\langle \mathrm{LM}_{<^{\mathrm{op}}}(S)\rangle = \langle \mathrm{LM}_{<^{\mathrm{op}}}(R)\rangle$.

For any $f \in \mathbb{K}[[x_1, \cdots, x_n]]$, consider $R_f := \{m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s) \in \mathrm{supp}(f) \mid m \in [x_1, \cdots, x_n] \;\; \wedge \;\; s \in R\} = \{m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s_i) \mid m \in [x_1, \cdots, x_n] \;\; \wedge \;\; i \in [\![1\,..\,r]\!]\}$.

Let $f \in \mathbb{K}[[x_1, \cdots, x_n]]$ and let us construct by induction a sequence $(f_k)_{k\in\mathbb{N}}$ of formal power series in $\mathbb{K}[[x_1, \cdots, x_n]]$.

Base step: $f_0 := f$.

Induction step: Let $k \in \mathbb{N}$ such that $f_0 \xrightarrow[S]{=} f_1 \xrightarrow[S]{=} \cdots \xrightarrow[S]{=} f_k$ and for all $i \in [\![0\,..\,k-1]\!]$, if $R_{f_i} \neq \varnothing \neq R_{f_{i+1}}$ then $\min_< R_{f_i} < \min_< R_{f_{i+1}}$.

If $R_{f_k} = \varnothing$, then $f_k \in \mathrm{NF}(\mathfrak{X})$ and $f = f_0 \xrightarrow[S]{\star} f_k$. If that is the case, define all subsequent $f_{k'}$ to be equal to f_k.

Otherwise, there exists a minimum element in R_{f_k}, denote it $m_k := \min_< R_{f_k}$. By definition, it means that we can rewrite m_k in f_k using a rule $s_i \in S$ (*i.e.* there exists $m \in [x_1, \cdots, x_n]$ such that $m_k = m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s_i)$). Thus define:

$$f_{k+1} := f_k - \frac{\langle f_k | m_k\rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s_i)} m \times s_i.$$

It follows that either $R_{f_{k+1}} = \varnothing$ or $\min_< R_{f_{k+1}} > \min_< R_{f_k}$, which allows us to continue the induction.

Limit step: in any case, repeating this process *ad infinitum* yields a infinite Cauchy sequence. Indeed, if at some point we have $R_{f_k} = \varnothing$, then the sequence is stationnary and therefore Cauchy. Otherwise, we have $\deg\left(\mathrm{LM}_{<^{\mathrm{op}}}(f_{k_1} - f_{k_2})\right) = \deg(m_{\min\{k_1,k_2\}})$ for any $k_1 \neq k_2$ in $\mathbb{N}$, therefore, since the order is compatible with the degree and by induction hypothesis, we get that:

$$\delta(f_{k_1}, f_{k_2}) = \frac{1}{2^{\deg(m_{\min\{k_1,k_2\}})}} \underset{k_1,k_2\to\infty}{\longrightarrow} 0.$$

Since $\mathbb{K}[[x_1, \cdots, x_n]]$ is a complete metric space, $(f_k)_{k\in\mathbb{N}}$ being Cauchy implies there exists $g \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $\lim_{k\in\mathbb{N}} f_k = g$. Hence, we obtain $f = f_0 \underset{S}{\overset{\circ}{\rightarrowtail}} g$ and, therefore, $f \underset{R}{\overset{\circ}{\rightarrowtail}} g$. But, by construction, g is a normal form. Hence the result. ☐

Lemma 3.11

If the order $<$ is *compatible with the degree*, then for all sequence $(f_k)_{k\in\mathbb{N}}$ of formal power series convering to $g \in \mathbb{K}[[x_1, \cdots, x_n]]$ and for all $m \in [x_1, \cdots, x_n]$, if for all $k \in \mathbb{N}$, we have $\mathrm{LM}_{<^{\mathrm{op}}}(f_k) \geqslant m$, then $\mathrm{LM}_{<^{\mathrm{op}}}(g) \geqslant m$.

Proof. Assume we have $\mathrm{LM}_{<^{\mathrm{op}}}(g) < m$, then for all $k \in \mathbb{N}$, $\mathrm{LM}_{<^{\mathrm{op}}}(g - f_k) < m$. Since the order is compatible with the degree, it follows that $\delta(g, f_k) > \frac{1}{2^{\deg(m)}}$. Hence, f_k cannot be converging to g, a contradiction. $\square$

Proposition 3.12

If the order $<$ is *compatible with the degree*, then the system $\mathcal{X}$ has globally attractive normal forms (GANF).

Proof. Let $\ell \in \mathrm{NF}(\mathcal{X})$. We start by showing that for all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \rightarrow\!\!\!\!\!\bigcirc\, g$ it follows that $\mathrm{LM}_{<^{\mathrm{op}}}(f - \ell) \leqslant \mathrm{LM}_{<^{\mathrm{op}}}(g - \ell)$. Let $m < \mathrm{LM}_{<^{\mathrm{op}}}(f - \ell)$. This means that $\langle f - \ell | m\rangle = 0$ from which we deduce $\langle f | m\rangle = \langle \ell | m\rangle$. Therefore, if $m \in \mathrm{supp}(f)$, then m is irreducible (since it also appears in ℓ). Since $f \rightarrow\!\!\!\!\!\bigcirc\, g$, there exists a sequence $(f_k)_{k\in\mathbb{N}}$ such that $f_0 = f$, $f \xrightarrow{\star} f_k$ for any $k \in \mathbb{N}$ and $\lim_{k\to\infty} f_k = g$. But, since all monomials $m < \mathrm{LM}_{<^{\mathrm{op}}}(f - \ell)$ that are in the support of f are irreducible, it follows that $\mathrm{LM}_{<^{\mathrm{op}}}(f - \ell) \leqslant \mathrm{LM}_{<^{\mathrm{op}}}(f_k - \ell)$ for all $k \in \mathbb{N}$. Hence by the previous lemma, we get the result.

Now we conclude to GANF by using the uniform structure provided by the metric δ and noticing that $\delta(f, \ell) \geqslant \delta(g, \ell)$ by compatibility with the degree. $\square$

Since a metric space is always Hausdorff, we get by Propositions 3.9-3.10-3.12 and Theorem 2.52, the following corollary:

Corollary 3.13

If the order $<$ is compatible with the degree, the following properties are equivalent for the system $\mathcal{X}_R$:

- Topological confluence,
- Infinitary confluence,
- Finitary confluence with respect to $\underset{R}{\overset{\star}{\succ\!\!\!\rightharpoonup\!\!\!\!\!\bigcirc}}$ or $\underset{R}{\overset{\star}{\succ\!\!\!\rightarrow\!\!\!\!\!\bigcirc}}$,
- Church-Rosser property with respect to $\underset{R}{\overset{\star}{\succ\!\!\!\rightharpoonup\!\!\!\!\!\bigcirc}}$, $\underset{R}{\overset{\star}{\succ\!\!\!\rightarrow\!\!\!\!\!\bigcirc}}$ or $\underset{R}{\rightarrow\!\!\!\!\!\bigcirc}$.
- Normal form (NF) property with respect to $\underset{R}{\succ\!\!\!\rightharpoonup\!\!\!\!\!\bigcirc}$, $\underset{R}{\succ\!\!\!\rightarrow\!\!\!\!\!\bigcirc}$ or $\underset{R}{\rightarrow\!\!\!\!\!\bigcirc}$,
- Unique normal form (UN) property with respect to $\underset{R}{\succ\!\!\!\rightharpoonup\!\!\!\!\!\bigcirc}$, $\underset{R}{\succ\!\!\!\rightarrow\!\!\!\!\!\bigcirc}$ or $\underset{R}{\rightarrow\!\!\!\!\!\bigcirc}$.

3.5 Standard bases

Definition 3.14 : Standard basis

Let I be an ideal of $\mathbb{K}[[x_1, \cdots, x_n]]$.
A subset R of $I \setminus \{0\}$ is said to be a **standard basis of I according to the order** $<$ if it satisfies:

$$\forall f \in I \setminus \{0\}, \quad \exists s \in R, \quad \exists m \in [x_1, \cdots, x_n], \quad \mathrm{LM}_{<^{\mathrm{op}}}(f) = m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s).$$

A set $R \subseteq \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\}$ is said to be **standard basis according to the order** $<$ if it a standard baiss of $I(R)$, the ideal generated by R, according to the order $<$.

Definition 3.15 : Standard representation

Let $R \subseteq \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\}$ and $<$ an admissible monomial order.
We say that $f \in \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\}$ admits a **standard representation with respect to** R **and** $<$ if there exists a finite subset $\{s_1, \cdots, s_r\} \subseteq R$ and a r-tuple $(q_1, \cdots, q_r) \in \mathbb{K}[[x_1, \cdots, x_n]]^r$ such that:

$$f = \sum_{i=1}^{r} q_i \times s_i \qquad \text{with} \qquad \mathrm{LM}_{<^{\mathrm{op}}}(f) = \min_{<} \{\mathrm{LM}_{<^{\mathrm{op}}}(q_i \times s_i) \mid i \in [\![1\,..\,r]\!], q_i \neq 0\}.$$

Note how if f has a standard representation with respect to R and $<$ then automatically we have $f \in I(R) \setminus \{0\}$. We will show that the converse implication is true if and only if R is a standard basis.

Fix $R \subseteq \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\}$ and $<$ an admissible monomial order on $[x_1, \cdots, x_n]$. Consider the topological rewriting system $\mathcal{X}_R := (\mathbb{K}[[x_1, \cdots, x_n]], \tau_\delta, \underset{R}{\rightarrow})$.

The following lemma follows in a straightforward manner from the definition of standard representation:

Lemma 3.16

Let $f \in \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\}$. If f admits a standard representation with respect to R and $<$, then it is in the ideal $I(R)$ generated by R and is lead-reducible by $\underset{R}{\rightarrow}$.

Lemma 3.17

Let $f \in \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\}$. If the order is compatible with the degree and if $f \underset{R}{\succ\!\!\!\to} 0$, then f admits a standard representation with respect to R and $<$.

Proof. Assume that $f \underset{R}{\succ\!\!\!\to} 0$. By definition, this means that there exists $S := \{s_1, \cdots, s_r\} \subseteq R$ and a sequence $(f_k)_{k\in\mathbb{N}}$ such that $f_0 = f$, $f_k \overset{=}{\underset{S}{\rightarrow}} f_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} f_k = 0$.

We are going to define inductively a sequence $\left(q_i^{(k)}\right)_{k\in\mathbb{N}}$ for each $i \in [\![1\,..\,r]\!]$ such that their limits form a standard representation of f with respect to S and $<$.

Base step: $q_i^{(0)} := 0$ for all $i \in [\![1\,..\,r]\!]$.

Inductive step: let $k \in \mathbb{N}$ and suppose that the r sequences $\left(q_i^{(k)}\right)_{k\in\mathbb{N}}$ are constructed up to and including the rank k in such a way that we have the following induction hypothesis:

$$f = f_k + \sum_{i=1}^{r} q_i^{(k)} \times s_i.$$

If $f_k = f_{k+1}$, then define $q_i^{(k+1)} := q_i^{(k)}$.

Otherwise, we know we must have $f_k \underset{S}{\rightarrow} f_{k+1}$. Hence, there exists $i_k \in [\![1\,..\,r]\!]$ and $m_k \in [x_1, \cdots, x_n]$ such that:

$$f_{k+1} = f_k - \frac{\langle f_k | m_k \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s_{i_k})\rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s_{i_k})} m_k \times s_{i_k}.$$

Then define $q_i^{(k+1)} := q_i^{(k)}$ for any $i \neq i_k$ and:

$$q_{i_k}^{(k+1)} := q_{i_k}^{(k)} + \frac{\langle f_k | m_k \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s_{i_k})\rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s_{i_k})} m_k.$$

By a simple computation and by making use of the induction hypothesis, we do indeed obtain the following equality:

$$f = f_{k+1} + \sum_{i=1}^{r} q_i^{(k+1)} \times s_i.$$

Thus, we can continue the induction.

Limit step: repeating this process *ad infinitum* this yields r sequences $\left(q_i^{(k)}\right)_{k\in\mathbb{N}}$. Note how, since $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence (because it converges), the sequence $(\deg(m_k))_{k\in\mathbb{N}}$ blows up to infinity. It follows that for each $i \in [\![1\,..\,r]\!]$, the sequence $\left(q_i^{(k)}\right)_{k\in\mathbb{N}}$ is Cauchy since it is either stationary or consist of terms differing by ever-bigger monomials from the $(m_k)_{k\in\mathbb{N}}$ sequence. Note $q_i^{(\infty)}$ the limit of $\left(q_i^{(k)}\right)_{k\in\mathbb{N}}$ for $i \in [\![1\,..\,r]\!]$. The induction hypothesis gives us at the limit:

$$f = \lim_{k\to\infty} \left(f_k + \sum_{i=1}^{r} q_i^{(k)} \times s_i \right) = 0 + \sum_{i=1}^{r} q_i^{(\infty)} \times s_i.$$

Moreover, the minimum $\min_< \left\{ \mathrm{LM}_{<^{\mathrm{op}}}\left(q_i^{(\infty)} \times s_i\right) \,\middle|\, i \in [\![1\,..\,r]\!], q_i \neq 0 \right\}$ is exactly the lowest monomial rewritten in the reduction sequence $f \underset{R}{\rightarrowtail\!\!\circledast} 0$ which is obviously $\mathrm{LM}_{<^{\mathrm{op}}}(f)$ because we rewrite into zero and thus the only way to rid ourselves of the $\mathrm{LM}_{<^{\mathrm{op}}}(f)$ is by rewriting it and its coefficient will never be affected by subsequent reductions.

In conclusion, the set $S = \{s_1, \cdots, s_r\} \subseteq R$ together with the r-tuple $\left(q_1^{(\infty)}, \cdots, q_r^{(\infty)}\right) \in \mathbb{K}[[x_1, \cdots, x_n]]^r$ form a standard representation for f with respect to R and $<$. □

Lemma 3.18

Let $f \in \mathbb{K}[[x_1, \cdots, x_n]]$ and $\alpha \in \mathrm{NF}(\mathfrak{X})$. Then, $f \rightarrowtail\!\!\circledast\, \alpha$ if, and only if, $f - \alpha \rightarrowtail\!\!\circledast\, 0$.

Proof. Assume $f \rightarrowtail\!\!\circledast\, \alpha$. Then there exists $(f_k)_{k\in\mathbb{N}}$ such that $f_0 = f$, $f_k \overset{=}{\longrightarrow} f_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} f_n = \alpha$. Consider the sequence $(f'_k)_{k\in\mathbb{N}}$ defined as $f'_k := f_k - \alpha$ for all $k \in \mathbb{N}$. Thus $f'_0 = f - \alpha$. For all $k \in \mathbb{N}$, if $f_k = f_{k+1}$, then $f'_{k+1} = f'_k$. Otherwise, $f_k \longrightarrow f_{k+1}$, hence there exists $m \in [x_1, \cdots, x_n]$ and $s \in R$ such that $\langle f_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\rangle \neq 0$ and $f_{k+1} = f_k - \frac{\langle f_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s$. But since α is a normal form, $\langle \alpha | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\rangle = 0$ and hence $\langle f_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\rangle = \langle f'_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\rangle$ and thus:

$$\begin{aligned} f'_{k+1} = f_{k+1} - \alpha &= \left(f_k - \frac{\langle f_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s \right) - \alpha \\ &= (f_k - \alpha) - \frac{\langle f_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s \\ f'_{k+1} &= f'_k - \frac{\langle f'_k | m \cdot \mathrm{LM}_{<^{\mathrm{op}}}(s)\rangle}{\mathrm{LC}_{<^{\mathrm{op}}}(s)} m \times s \end{aligned}$$

Therefore, in either case, we have $f'_k \overset{=}{\longrightarrow} f'_{k+1}$. Finally, we have $\lim_{k\to\infty} f'_k = (\lim_{k\to\infty} f_k) - \alpha = \alpha - \alpha = 0$. Hence $f - \alpha \rightarrowtail\!\!\circledast\, 0$.

Assume that $f - \alpha \rightarrowtail\!\!\circledast\, 0$. Hence, by Corollary 3.4 of translation, there exists $g \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \rightarrowtail\!\!\circledast\, g$ and $\alpha \rightarrowtail\!\!\circledast\, g$. Since this implies $\alpha \,-\!\!\circledast\, g$ and we also have that $\alpha \in \mathrm{NF}(\mathfrak{X})$, it follows from Proposition 2.26 that $\alpha = g$ and hence $f \rightarrowtail\!\!\circledast\, g = \alpha$. □

Theorem 3.19 : (Characterisation of standard bases)

Let $R \subseteq \mathbb{K}[[x_1, \cdots, x_n]] \setminus \{0\}$ and $<$ an admissible monomial order. Consider the topological rewriting system $\mathfrak{X}_R := (\mathbb{K}[[x_1, \cdots, x_n]], \tau_\delta, \underset{R}{\longrightarrow})$. Denote by $I := I(R)$ the ideal generated by R.

Then, if the order $<$ is compatible with the degree, all the following statements are equivalent:

(i) the system $\mathcal{X}_R$ is *topologically confluent*,

(ii) for all $f \in I$, we have $f \underset{R}{\rightarrowtail} 0$,

(iii) for all $f \in I$, we have $f \underset{R}{\rightarrowtail} 0$,

(iv) for all $f \in I$, we have $f \longrightarrow 0$,

(v) for all $f \in I \setminus \{0\}$, then f admits a standard representation with respect to R and $<$,

(vi) for all $f \in I \setminus \{0\}$, then f is reducible,

(vii) the set R is a *standard basis*,

(viii) the map $\begin{array}{rcl} \mathrm{NF}(\mathcal{X}) & \to & \mathbb{K}[[x_1,\cdots,x_n]]/I \\ \alpha & \mapsto & \alpha + I \end{array}$ is bijective.

Proof. (i) $\Rightarrow$ (ii) and (iii) and (iv): By Corollary 3.13, the assumption (i) implies the normal form (NF) properties with respect to $\underset{R}{\rightarrowtail}$, $\underset{R}{\rightarrowtail}$ and $\underset{R}{\longrightarrow}$. Let $f \in I$. Then $f - 0 \in I$, that is to say: $f \equiv 0 \pmod{I}$. By Theorem 3.8, it follows that $f \overset{\star}{\underset{R}{\leftrightarrow}} 0$, $f \overset{\star}{\underset{R}{\leftrightarrow}} 0$ and $f \overset{\star}{\underset{R}{\longleftrightarrow}} 0$. Finally, by NF, since 0 is a normal form and the base relations are transitive, we obtain that $f \underset{R}{\rightarrowtail} 0$, $f \underset{R}{\rightarrowtail} 0$ and $f \underset{R}{\longrightarrow} 0$.

(ii) $\Rightarrow$ (v): This exactly the content of Lemma 3.17.

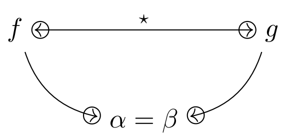
(iii) or (iv) $\Rightarrow$ (vi): By contradiction, suppose (iii) or (iv) and $\neg$ (vi), let $f \in I \setminus \{0\}$. By assumption $\neg$ (vi) we get that f is not reducible, *i.e.* $f \in \mathrm{NF}(\mathcal{X}_R)$. But by assumption (iii) or (iv), we have $f \underset{R}{\longrightarrow} 0$. Hence, by Proposition 2.26, it follows that $f = 0$, a contradiction.

(v) $\Rightarrow$ (vi): This is evident with Lemma 3.16.

(vi) $\Rightarrow$ (vii): By contradiction, assume (vi) and that there exists $f \in I \setminus \{0\}$ such that $\mathrm{LM}_{<^{\mathrm{op}}}(f)$ is not divisible by any $\mathrm{LM}_{<^{\mathrm{op}}}(s)$ for $s \in R$. By Proposition 3.10, there exists $\alpha \in \mathrm{NF}(\mathcal{X}_R)$ such that $f \underset{R}{\longrightarrow} \alpha$. Hence, by Lemma 3.6, we have $f - \alpha \in I$ and $f \neq \alpha$ since $f \in I$ is reducible by assumption (vi). Hence, we get that $\alpha \in I \setminus \{0\}$, which means by assumption (vi) again, that α is reducible, a contradiction.

(vii) $\Rightarrow$ (viii): By contradiction, suppose the map is not injective. We thus have $\alpha, \beta \in \mathrm{NF}(\mathcal{X}_R)$ such that $\alpha \neq \beta$ and $\alpha + I = \beta + I$. It follows that $\alpha - \beta \in I \setminus \{0\}$. By assumption (iv), there exists $s \in R$ such that $\mathrm{LM}_{<^{\mathrm{op}}}(s)$ divides $\mathrm{LM}_{<^{\mathrm{op}}}(\alpha - \beta)$. This necessarily means that at least one of $\mathrm{LM}_{<^{\mathrm{op}}}(\alpha)$ and $\mathrm{LM}_{<^{\mathrm{op}}}(\beta)$ is reducible which contradicts α or β being normal forms.

(viii) $\Rightarrow$ (i): By Corollary 3.13, showing topological confluence of $\mathcal{X}_R$ is equivlaent to showing the topological Church-Rosser property. Let $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \overset{\star}{\longleftrightarrow} g$. By Proposition 3.10, there exist $\alpha, \beta \in \mathrm{NF}(\mathcal{X}_R)$ such that $f \longrightarrow \alpha$ and $g \longrightarrow \beta$. Then, $\alpha \overset{\star}{\longleftrightarrow} \beta$. By Theorem 3.8, it follows that $\alpha - \beta \in I$. By assumption (viii), we get that $\alpha = \beta$. Hence, we obtain:



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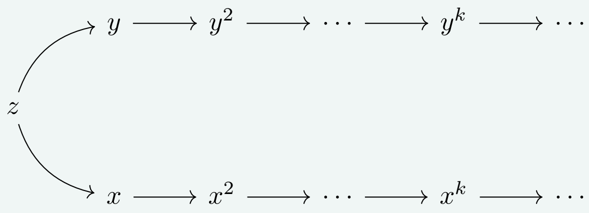
Example 3.20 : (Counter-example "topological confluence $\Rightarrow$ confluence")

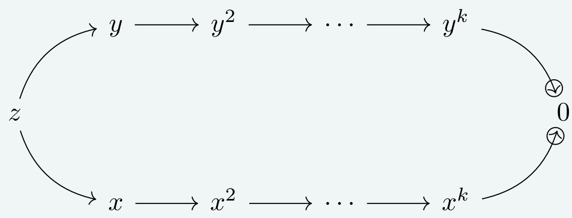
Consider the $\mathbb{K}[[x,y,z]]$, $<$ the deglex monomial order such that $z < y < x$ and:

$$R = \left\{z - x, z - y, y - y^2, x - x^2\right\}.$$

Notice how R is a standard basis according to $<$. Indeed, $\mathrm{LM}_{<^{\mathrm{op}}}(R) = \{z, y, x\}$ and for any $f \in I(R)$, we have $\mathrm{val}(f) \geqslant 1$, hence by compatibiliy with the degree, it follows that there exists $\xi \in \{z, y, x\}$ such that $\mathrm{LM}_{<^{\mathrm{op}}}(f)$ is divisible by ξ. By the previous theorem, we conclude that the topological rewriting system associated to $<$ and R is topologically confluent.
However, consider the following branching:



The two branches will never join in a finite amount of steps, therefore the system cannot be confluent.
Note how by topological confluent we can join the branches topologically. Indeed, notice how the two branches converge to 0 for the topology, therefore we obtain:



3.6 Existence of chains for formal power series

Throughout this subsection fix n a positive integer, $\mathbb{K}$ a (commutative) field, $<$ an admissible monomial order on $[x_1, \cdots, x_n]$ and R a non-empty set of non-zero formal power series in $\mathbb{K}[[x_1, \cdots, x_n]]$. Write $\mathfrak{X} := (X, \tau_\delta, \longrightarrow)$ the topological rewriting system where $X := \mathbb{K}[[x_1, \cdots, x_n]]$ and $\longrightarrow := \underset{R}{\longrightarrow}$.

Theorem 3.21

If the order $<$ is compatible with the degree, then *chains exist* in $\mathfrak{X}$ if and only if for all $f, g \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \underset{\lim}{\longrightarrow} g$ there exists $h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \longrightarrow h \longrightarrow\!\!\!\!\!\circ\; g$.

Proof. The left-to-right direction is trivially verified considering the Proposition **??**.
To prove the other direction, we make use of the fact that, $\mathbb{K}[[x_1, \cdots, x_n]]$ being a metric space, it is first-countable. Therefore, we can use Theorem **??**. Let us fix the neighbourhood bases for proxmity maps to be, for $g \in \mathbb{K}[[x_1, \cdots, x_n]]$, $\mathcal{B}_n^g := \overline{B}\left(g, \frac{1}{2^n}\right)$. Therefore, by compatibility with the degree, the proximity maps are such that:

$$P_g(f) = \begin{cases} \infty & \text{if } f = g, \\ \deg(\mathrm{LM}_{<^{\mathrm{op}}}(f - g)) & \text{otherwise.} \end{cases}$$

Hence, for chains to exist it suffices to show that there exists $h \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \overset{\star}{\longrightarrow} h \longrightarrow\!\!\!\!\oplus\, g$ and $\deg(\mathrm{LM}_{<^{\mathrm{op}}}(h-g)) > \deg(\mathrm{LM}_{<^{\mathrm{op}}}(f-g))$.

Assume the right-hand side of the equivalence. Let f and g in $\mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \longrightarrow\!\!\!\!\oplus\, g$. If $f \overset{\star}{\longrightarrow} g$ then trivially $f \rightarrowtail\!\!\!\!\oplus\, g$. Let us suppose then that $f \underset{\lim}{\longrightarrow} g$. By hypothesis, this means that there exists $h_0 \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $f \longrightarrow h_0 \longrightarrow\!\!\!\!\oplus\, g$. Now, necessarily $h_0 \underset{\lim}{\longrightarrow} g$ because if $h_0 \overset{\star}{\longrightarrow} g$ then we contradict $f \underset{\lim}{\longrightarrow} g$. Therefore we can apply the hypothesis again on $h_0 \underset{\lim}{\longrightarrow} g$ and find $h_1 \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $h_0 \longrightarrow h_1 \underset{\lim}{\longrightarrow} g$. And so on and so fourth, we construct a sequence $(h_k)_{k\in\mathbb{N}}$ such that $h_k \longrightarrow h_{k+1}$ and $h_k \underset{\lim}{\longrightarrow} g$ for all $k \in \mathbb{N}$.

Now, since $f \underset{\lim}{\longrightarrow} g$, we have that, for all $\ell \in \mathbb{K}[[x_1, \cdots, x_n]]$ such that $\ell \underset{\lim}{\longrightarrow} g$, ℓ and f agree on all monomials that are strictly smaller that the minimum between $\mathrm{LM}_{<^{\mathrm{op}}}(\ell - g)$ and $\mathrm{LM}_{<^{\mathrm{op}}}(f-g)$. Indeed: write $p_{\min} \in \{f, \ell\}$ the formal power series such that $\mathrm{LM}_{<^{\mathrm{op}}}(p_{\min} - g) = \min_{<}\{\mathrm{LM}_{<^{\mathrm{op}}}(f-g), \mathrm{LM}_{<^{\mathrm{op}}}(\ell-g)\}$; assume that there exists $m_0 \in [x_1, \cdots, x_n]$ such that $m_0 < \mathrm{LM}_{<^{\mathrm{op}}}(p_{\min} - g)$ and $\langle f|m_0\rangle \neq \langle \ell|m_0\rangle$. By the first assumption, we get that $\langle f-g|m_0\rangle = \langle \ell - g|m_0\rangle = 0$ from which we deduce that $\langle f|m_0\rangle = \langle \ell|m_0\rangle$, a contradiction of the second assumption.

Now, since we have finitely many variables and the order is compatible with the degree, there are only finitely many monomials m such that $m \leqslant \mathrm{LM}_{<^{\mathrm{op}}}(\ell)$ for any fixed non-zero $\ell \in \mathbb{K}[[x_1, \cdots, x_n]]$. Therefore, the infinite sequence $(h_k)_{k\in\mathbb{N}}$ cannot verify that $\mathrm{LM}_{<^{\mathrm{op}}}(h_0 - g) \leqslant \mathrm{LM}_{<^{\mathrm{op}}}(f-g)$ because otherwise this would mean we have rewritten in f a monomial m smaller than $\mathrm{LM}_{<^{\mathrm{op}}}(f-g)$ (thus, $\langle f|m\rangle = \langle g|m\rangle$) and so, we would have $\langle h_0|m\rangle = 0$. However, we must have $h_0 \underset{\lim}{\longrightarrow} g$ so we must be able to "recover" the correct coeffient of m which is possible only by rewriting monomials that are smaller than m. But this means that we have the same problem for those smaller monomials that we rewrite, and since they are finitely many of them and that the sequence $(h_k)_{k\in\mathbb{N}}$ is infinite, this is impossible.

Finally, by repeating this reasoning on the sequence $(h_k)_{k\in\mathbb{N}}$, we get that:

$$\mathrm{LM}_{<^{\mathrm{op}}}(f-g) < \mathrm{LM}_{<^{\mathrm{op}}}(h_0 - g) < \mathrm{LM}_{<^{\mathrm{op}}}(h_1 - g) < \cdots.$$

Hence, since we have finitely many variables and the order is compatible with the degree, it follows that there necessarily exists a $k \in \mathbb{N}$ such that $P_g(h_k) = \deg(\mathrm{LM}_{<^{\mathrm{op}}}(h_k - g)) > \deg(\mathrm{LM}_{<^{\mathrm{op}}}(f-g)) = P_g(f)$, which concludes the proof that chains exist. □

Example 3.22 : (Chains do not always exist for commutative formal power series)

Consider $\mathbb{K}[[x, y]]$ with the usual adic topology, $<$ the deglex monomial order with $x < y$ and the relation defined by:

$$R := \{x - y^k \mid k \geqslant 1\}.$$

Then, we can see that $x \longrightarrow\!\!\!\!\oplus\, 0$ because $x \longrightarrow y^k$ for any $k \geqslant 1$ and $\lim_{k\in\mathbb{N}} y^k = 0$.
However, we do not have any rewriting chains from x to 0 (*i.e.* we do not have $x \rightarrowtail\!\!\!\!\oplus\, 0$) because each y^k $(k \geqslant 1)$ is a normal form in the system.

Theorem 3.23 : (CHAINS CONJECTURE)

Here is a following conjecture that remains to be proven or disproven:
If the order $<$ is compatible with the degree and R is **finite**, then chains exist in the system.

Todo: show special cases that have been proven (if R is a standard basis, if the formal power series considered are actually polynomials, etc.).

References