Introduction to mathematics

Part III: Elements of set theory

Contents

1 Introduction

Set theory is a mathematical theory that study "sets", their properties and the operations we can apply on them. We will not be taking an axiomatic approach nor will we investigate some of the deeper concerns that arise from such formal developments of set theory. We simply wish to introduce the basic definitions and properties to the beginner reader.

"To formalise" means to define unambiguously new concepts using preexisting definitions that are either other formalised concepts or primitive notions of the theory. For this reason, set theory is central to some, if not all, of math. Indeed set theory is *the* standard framework used to formalise everything in mathematics. It is thus of significant importance that the basic elements of set theory are understood for anyone who wishes to study mathematics. This is the objective of this document, as well as the following document in the series "Introduction to mathematics".

For the time being, in this document, we will talk about the primitive notion of sets, what are subsets, what it means to use intentional set definitions, define some of the basic operations we can apply on sets and finally, we will mention the well-known sets of numbers to illustrate what we will have introduced by then.

The prerequisite knowledge to follow this document is the basics of mathematical logic including what are propositions, logical operators, predicates, quantifiers... These notions are discussed in the document "part II" of this series.

For a more in-depth but yet very easy-to-read introduction to set theory see [Halmos, 1960].

2 Sets

No formal definition of what a set is can be given: it is such a primitive concept that it cannot really be formalised *i.e.* be defined in terms of some preexisting notions. This is why most of the time an intuitive definition is provided rather than a formal one.

Definition 2.1 : Set

A **set** is a mathematical object that can be viewed as a collection of mathematical objects. Here, mathematical objects refer to any well-defined entities such as numbers, functions or even sets.

The term **collection** is often used as a synonym for set. The word **class** is also sometimes used that way but, since some developments of set theory coin that word for another specific meaning, it is not recommended to use it as much. By convention, sets are denoted by capital Latin letters.

If A is a set, then the mathematical objects that A contains are called the **elements** of A. If a is an element of the set A, then we write $a \in A$ which is to be read as "a is in A" or "a is an element of A" or even "a belongs to A". This symbol $\in$ is used to represent the **membership relation** of an element with its containing set. If we want to denote that some particular mathematical object b is not in the set A then we would write $b \notin A$.

As we have discussed in part II, variables are said to take values from a certain domain of discourse. In practice this domain of discourse is not explicitly defined and rather than having quantifiers range over the whole domain of discourse we restrict them to some sets we explicitly specify. Concretely, if $P[x]$ is a predicate (recall that the notation $P[x]$ means that if P has a free variable then it is x), rather than having $\forall x, P(x)$ (that is, having the variable range over the whole domain of discourse) we specify the set A that contains the possible values for x and we write: $\forall x \in A, P(x)$. This is to be understood as a shorthand for:

$$(\forall x \in A, P(x)) \overset{\text{def}}{\Leftrightarrow} (\forall x, x \in A \Rightarrow P(x))$$

Similarly, we define:

$$(\exists x \in A, P(x)) \overset{\text{def}}{\Leftrightarrow} (\exists x, x \in A \land P(x))$$

Here the symbol $\overset{\text{def}}{\Leftrightarrow}$ means that the left hand side of the equivalence is being defined as a shorthand for the right hand side of the equivalence.

Then we can show that the negations of those propositions are given by:

$$\begin{aligned}
\neg(\forall x \in A, P(x)) &\Leftrightarrow \neg(\forall x, x \in A \Rightarrow P(x)) && \text{by definition of "}\forall x \in A\text{"}\\
&\Leftrightarrow (\forall x, x \notin A \lor \neg P(x)) && \text{since } (X \Rightarrow Y) \Leftrightarrow (\neg X \lor Y) \text{ tautology}\\
&\Leftrightarrow (\exists x, x \in A \land \neg P(x)) && \text{by negating } (\forall \to \exists, \text{ and De Morgan laws})\\
\neg(\forall x \in A, P(x)) &\Leftrightarrow (\exists x \in A, \neg P(x)) && \text{by definition of "}\exists x \in A\text{"}
\end{aligned}$$

and

$$\begin{aligned}
\neg(\exists x \in A, P(x)) &\Leftrightarrow \neg(\exists x, x \in A \land P(x)) && \text{by definition of "}\exists x \in A\text{"}\\
&\Leftrightarrow (\forall x, x \notin A \lor \neg P(x)) && \text{by negating } (\exists \to \forall, \text{ and De Morgan laws})\\
&\Leftrightarrow (\forall x, x \in A \Rightarrow \neg P(x)) && \text{since } (X \Rightarrow Y) \Leftrightarrow (\neg X \lor Y) \text{ tautology}\\
\neg(\exists x \in A, P(x)) &\Leftrightarrow (\forall x \in A, \neg P(x)) && \text{by definition of "}\forall x \in A\text{"}
\end{aligned}$$

When we want to shorten the successions of quantifiers acting on variables drawn from the same set we use the following notation:

$$\forall x, y \in A, P(x,y) \overset{\text{def}}{\Leftrightarrow} \forall x \in A, \forall y \in A, P(x,y)$$

where $P[x,y]$ is a predicate.

In this example we only use two variables but it generalises to any number of variables:

$$\forall x_1, x_2, \cdots, x_n \in A, P(x_1, x_2, \cdots, x_n) \overset{\text{def}}{\Leftrightarrow} \forall x_1 \in A, \forall x_2 \in A, \cdots, \forall x_n \in A, P(x_1, x_2, \cdots, x_n)$$

Now that we have introduced several notations about sets, let us talk about how we can construct sets. To *define* a set we can go one of two ways: an extensional definition or an intentional definition. We will talk about the latter later in the document, for now let us explain what the former means.

The **extensional definition** of a set A consists in listing explicitly all of the elements of the set A. One handy way to do that is by using the **Roster notation**: consider the set A whose elements are the integers 0, 2 and 5, in Roster notation we would write:

$$A = \{0, 2, 5\}$$

It consists of listing the elements of the set separated by commas and enclosed within curly braces.

Defining sets with finitely many elements is made really easy by this notation, especially if there are only a few elements. However, we are considerably limited to what sets we can define in that manner. One way to circumvent this limitation is by adding some implicit meaning with the use of ellipsis. For instance if we want to define the set B as the set of integers from 1 to 100 we could write:

$$B = \{1, 2, \cdots, 100\}$$

Now if we want the set C to contain all the integers from 0 (thus it would have infinitely many elements) we could write:

$$C = \{0, 1, 2, \cdots\}$$

In a similar fashion we can define a lot of sets with the aid of ellipsis when there is a logical connection between the first few elements. Nevertheless, it is important to always keep in mind that ellipsis give an implicit meaning that could be interpreted in different ways, which is exactly what we want to avoid in math. If ever there is a risk for confusion, then it is advised to explain, in a natural language such as English, what the set consists of or to use an intentional definition.

One important thing to remember is: sets are completely determined by their elements. This means two sets are said to be **equal if and only if they consist of exactly the same elements**. Formally, if A and B are sets, then:

$$A = B \;\Leftrightarrow\; (\forall x, x \in A \Leftrightarrow x \in B)$$

This implies that the order in which the elements are presented in Roster notation does not change the underlying set. For example we have:

$$\{0, 2, 5\} = \{5, 0, 2\}$$

Also the elements are *unique* in the sense that a same element only actually "appear" once in the set no matter how many times it appears in the Roster notation. For example we have:

$$\{0, 0, 0\} = \{0\}$$

To denote that two sets A and B are not equal we use the notation $A \neq B$. By definition, two sets are not equal if there is an element in one that is not in the other.

One notable set is the **empty set**, denoted by the symbol $\varnothing$: the set with no elements. In Roster notation, we would have:

$$\varnothing = \{\}$$

Note that for any predicate $P[x]$, the proposition $\exists x \in \varnothing, P(x)$ is always false because there exists no element in the set $\varnothing$, so no element can satisfy the proposition. Considering now the proposition $\forall x \in \varnothing, P(x)$. This is always true and to understand why, let us think about what it would mean to deny that proposition: its negation is $\exists x \in \varnothing, \neg P(x)$ but we have just mentioned that a proposition that consists of $\exists x \in \varnothing$ is always false. The negation being false necessarily implies that the original proposition is true. So to recap, a proposition of the form "$\exists x \in \varnothing, P(x)$" is always false while a proposition of the form "$\forall x \in \varnothing, P(x)$" is always true.

We call **singleton** any set that consists of exactly one element. An **unordered pair** (or simply **pair**) is a set containing exactly two distinct elements.

3 Subsets

As it is often the custom in mathematics, after defining a new object we are interested in knowing if it is possible to "extract" from the new object another object verifying similar properties as the containing object. In terms of sets, we ask ourselves: can we obtain new sets by extracting from preexisting sets? The answer is yes. Since sets are just collection of elements, it suffices to take some of the elements of the containing set to create a new set that we will call "subset".

Definition 3.1 : Subset

Let A be a set. We say that a set B is a **subset** of A if every element of B is also in A. Alternatively, we also say that A is a **superset** of B, or that B is **contained** in A.
We denote that relation by writing $B \subseteq A$ or, equivalently, $A \supseteq B$. Formally:

$$B \subseteq A \overset{\text{def}}{\Leftrightarrow} \forall x \in B, x \in A$$

The symbol $\subseteq$ is said to represent the **inclusion** relation. To denote that B is not a subset of A we write $B \nsubseteq A$ or, equivalently, $A \nsupseteq B$.

For instance, if we consider the set $A = \{a, b, c\}$ then the sets $\{a, b\}$ and $\{c\}$ are examples of subsets of A.

Let us note two important inclusion relations verified by any set:

Proposition 3.1

Let A be a set. Then: $\varnothing \subseteq A$ and $A \subseteq A$.

Proof.
- Showing that $\varnothing \subseteq A$ means showing $\forall x \in \varnothing, x \in A$. But we have already discussed that any proposition of the form $\forall x \in \varnothing, P(x)$ is always true. Thus, we do have $\varnothing \subseteq A$.
- Let $a \in A$, then $a \in A$ which by definition means $A \subseteq A$. $\square$

Sometimes, we wish to work with collections of subsets from a given set A. We thus introduce the following set containing all the subsets of A:

Definition 3.2 : Power set

Let A be a set. The **power set** of A is the collection of all the subsets of A.
We denote it as $\mathcal{P}(A)$ or 2^A. Formally:

$$X \in \mathcal{P}(A) \overset{\text{def}}{\Leftrightarrow} X \subseteq A$$

For example, if $A = \{a, b, c\}$ consists of three elements a, b and c then the power set is given by:

$$\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$$

It is important to understand that the power set is a set whose elements are also sets. Note for instance that $\{a\} \in \mathcal{P}(A)$ but *a priori* $a \notin \mathcal{P}(A)$: the elements are themselves sets containing the elements of the original set. Note as well that, no matter the set A considered, the power set $\mathcal{P}(A)$ is not empty as it contains at least the empty set $\varnothing$.

There exist notations for the inclusion relation $\subseteq$ analogous to the ones for the membership relation $\in$. Let $P[X]$ be a predicate and A be a set. We introduce the notations:

- for the universal quantifier: $\forall X \subseteq A, P(X) \overset{\text{def}}{\Leftrightarrow} \forall X \in \mathcal{P}(X), P(X)$
- for the existential quantifier: $\exists X \subseteq A, P(X) \overset{\text{def}}{\Leftrightarrow} \exists X \in \mathcal{P}(X), P(X)$

Here is a very important theorem that characterises the equality between sets:

Theorem 3.1

Let A and B be two sets.

$$A = B \ \Leftrightarrow\ (A \subseteq B \ \wedge\ B \subseteq A)$$

Proof. Since we want to prove a logical equivalence we want to show a double implication.

- Let us start by assuming that $A = B$. Then by proposition 3.1 we have $A \subseteq B$ and $B \subseteq A$.
- Now assume that both $A \subseteq B$ and $B \subseteq A$. Then if x is an element of the domain of discourse, if we assume that $x \in A$ it follows that $x \in B$ since $A \subseteq B$, which means $x \in A \Rightarrow x \in B$. And similarly, if we assume $x \in B$ we have $x \in A$ by $B \subseteq A$ and so $x \in B \Rightarrow x \in A$. Thus, we have $x \in A \Leftrightarrow x \in B$ for all x which enables us to conclude that $A = B$. □

We will call this method of showing an equality between sets: **double inclusion**. In practice this is usually what we use to prove that two sets are equal.

This second relation $A \subseteq A$ from proposition 3.1 means that the inclusion relation is *reflexive*. The theorem 3.1 means that the inclusion relation is *antisymmetric*. Now the following proposition states that the inclusion relation is *transitive* (Those terms will be defined in a more general context in part IV when we will discuss relations)

Proposition 3.2

Let A, B and C be sets. Then: $(A \subseteq B \ \wedge\ B \subseteq C) \ \Rightarrow\ A \subseteq C$.

Proof. Assume $A \subseteq B$ and $B \subseteq C$. Let $x \in A$. Then $x \in B$ for $A \subseteq B$. But, since $B \subseteq C$, this implies that $x \in C$. Thus $A \subseteq C$. □

As stated in theorem 3.1, sets are said to be equal if they verify a double inclusion. Sometimes though, one inclusion is satisfied while the other is not. To denote when this is the case, we can define another relation called **strict inclusion**:

Definition 3.3 : Strict subset

Let A and B be two sets. Then we denote that B is a **proper subset** of A or that A is a **proper superset** of B by writing $B \subsetneq A$ or $A \supsetneq B$. Formally defined as:

$$B \subsetneq A \overset{\text{def}}{\Leftrightarrow} (B \subseteq A \ \wedge\ B \not\supseteq A)$$

Note that the symbol $\subset$ also exists and would suggest that it is used to represent the strict version of the inclusion relation. However, in practice, most authors use it as a replacement for $\subseteq$ and not $\subsetneq$.

4 Intentional set definition

Let us now talk about the other way to define sets. An **intentional definition** of a set B consists in specifying the logical condition objects have to satisfy in order to be elements of the set B. To avoid problems with potential paradoxes that could arise otherwise, we will restrict ourselves to use intentional definitions for sets only as subsets of preexisting sets. In other words, to define intentionally a set we need both the logical condition on its elements *and* the superset from which the elements are drawn.

We use the **set-builder notation** to define intentionally sets. Let A be a set and $P[x]$ a predicate. Then we define the subset B, of the set A, whose elements are exactly those which satisfy the predicate $P[x]$ by writing:

$$B = \{x \in A \mid P(x)\}$$

In other words:

$$\forall x, \quad (x \in B \overset{\text{def}}{\Leftrightarrow} (x \in A \ \wedge \ P(x)))$$

(Note that the set-builder notation acts as a variable-binding operator on the variable x. Thus the symbol x can be replaced by any other symbol without altering the meaning, as long as that symbol does not already appear in the expression.)

Alternative notations include replacing the vertical bar "|" by a comma ",", by a semi-colon ";", by a colon ":" or by a forward slash "/" but they all mean the same.

We deduce from the definition above that for an element x of the domain of discourse we have the equivalences:

$$x \notin B \ \Leftrightarrow \ (x \notin A \vee \neg P(x)) \ \Leftrightarrow \ (x \in A \Rightarrow \neg P(x))$$

We have that $B = A$, if and only if, the proposition $P(x)$ is verified for all $x \in A$. In particular, if $P(x)$ is a tautology for any $x \in A$ then $A = B$. On the contrary, if $P(x)$ is a contradiction for all $x \in A$, then B is empty.

We call the predicate $P[x]$ the **admissibility condition** of the set B: the elements of B are exactly the elements of A that satisfy the admissibility condition of B.

The following proposition establishes what it means for intentionally defined sets to be subset of one another with respect to their admissibility condition.

Proposition 4.1

Let A and B be two sets. Let $P[x]$ and $Q[x]$ be two predicates.
Let $X = \{x \in A \mid P(x)\}$ and $Y = \{x \in B \mid Q(x)\}$. Then:

$$X \subseteq Y \ \Leftrightarrow \ (\forall x \in A, P(x) \Rightarrow (x \in B \wedge Q(x)))$$

Proof. ▪ Assume $X \subseteq Y$. Let $x \in A$. Suppose $P(x)$ is true. Then $x \in X$ by definition of a set defined intentionally. But since $X \subseteq Y$, it follows that $x \in Y$. This means that $Q(x)$ is also true as well as $x \in B$. Thus we have shown that for all $x \in A$, $P(x)$ true implies ($Q(x)$ true and $x \in B$).

▪ Assume that $\forall x \in A, P(x) \Rightarrow (x \in B \wedge Q(x))$. Let $x \in X$. Then $P(x)$ is necessarily true. But since $P(x) \Rightarrow (x \in B \wedge Q(x))$, by modus ponens it follows that $x \in B$ and that $Q(x)$ is true and thus $x \in Y$. So we have $X \subseteq Y$. □

Corollary 4.1

With the same notations, in the very common case where $A = B$, we have:

$$X \subseteq Y \ \Leftrightarrow \ (\forall x \in A, P(x) \Rightarrow Q(x))$$

What one must remember from this proposition and its corollary is that when two sets are defined intentionally then the inclusion relation is characterised by the logical implication of the admissibility conditions.

A consequence of this is another characterisation for the equality of sets, when they are intentionally defined:

Proposition 4.2

Let A and B be two sets. Let $P[x]$ and $Q[x]$ be two predicates.
Let $X = \{x \in A \mid P(x)\}$ and $Y = \{x \in B \mid Q(x)\}$. Then:

$$X = Y \quad \Leftrightarrow \quad (\forall x, (x \in A \wedge P(x)) \Leftrightarrow (x \in B \wedge Q(x)))$$

Proof. Proposition 4.1 gives us the equivalence $X \subseteq Y \Leftrightarrow (\forall x \in A, P(x) \Rightarrow (x \in B \wedge Q(x)))$.
But we know that $(M \Rightarrow (N \Rightarrow R)) \Leftrightarrow ((M \wedge N) \Rightarrow R)$ is a tautology.
Then $X \subseteq Y \Leftrightarrow (\forall x, (x \in A \wedge P(x)) \Rightarrow (x \in B \wedge Q(x)))$.
Similarly, we obtain $Y \subseteq X \Leftrightarrow (\forall x, (x \in B \wedge Q(x)) \Rightarrow (x \in A \wedge P(x)))$.
Thus, by theorem 3.1, we have $X = Y \Leftrightarrow (\forall x, (x \in A \wedge P(x)) \Leftrightarrow (x \in B \wedge Q(x)))$ □

Corollary 4.2

With the same notations, in the very common case where $A = B$, we have:

$$X = Y \quad \Leftrightarrow \quad (\forall x \in A, P(x) \Leftrightarrow Q(x))$$

We said earlier in the document that, in practice, we generally use a double inclusion to prove the equality of two sets. The second most usual way to prove the equality between two sets, that have been introduced by an intentional definition, is by showing that their admissibility conditions are logically equivalent.

5 Operations on sets

We have now the ability to define sets in two different manners: with an extensional definition and with an intentional definition. Let us now introduce other ways to construct new sets from preexisting ones: by means of what is called set operations. A lot of set operations exist; we are going to define the most important ones, starting with the union of two sets:

Definition 5.1 : Union

Let A and B be two sets. The **union** of A and B is the set, denoted $A \cup B$, defined as:

$$x \in (A \cup B) \overset{\text{def}}{\Leftrightarrow} (x \in A \quad \vee \quad x \in B)$$

For example, if :

- $A := \{1, 2, 3\}$ and $B := \{4, 5\}$ then $A \cup B = \{1, 2, 3, 4, 5\}$.
- $A := \{1, 2, 3\}$ and $B := \{3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$.
- A is a set, then $A \cup \varnothing = A$.

We can define another operation, called the intersection for two sets:

Definition 5.2 : Intersection

Let A and B be two sets. The **intersection** of A and B is the set, denoted $A \cap B$, defined as:

$$x \in (A \cap B) \overset{\text{def}}{\Leftrightarrow} (x \in A \quad \wedge \quad x \in B)$$

For example, if:

- $A := \{1, 2, 3\}$ and $B := \{4, 5\}$ then $A \cap B = \varnothing$.
- $A := \{1, 2, 3\}$ and $B := \{3, 4\}$ then $A \cap B = \{3\}$.
- A is a set, then $A \cap \varnothing = \varnothing$.

Now some elementary consequences from the definitions of union and intersection of sets:

Proposition 5.1

Let A, B and C be sets. Then we have:

$$\begin{aligned} A \cup A &= A & A \cap A &= A \\ A \cup B &= B \cup A & A \cap B &= B \cap A \\ A \cup (B \cup C) &= (A \cup B) \cup C & A \cap (B \cap C) &= (A \cap B) \cap C \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) & A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

Proof. These equalities are direct consequences of tautologies we have mentioned in part II regarding the logical analogous of $\cup$ and $\cap$: $\vee$ for $\cup$ and $\wedge$ for $\cap$. $\square$

The third of those equations in each column provides a meaning to the expressions "$A \cup B \cup C$" and "$A \cap B \cap C$" (note the absence of parentheses) since it means that the order in which the operations are performed does not impact the resulting set: this property is called *associativity*. This property generalises to any finite number of sets: it thus enables us to define without ambiguities the union and the intersection for any arbitrary number n of sets $A_1, A_2, \cdots, A_n$:

$$\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup \cdots \cup A_n \qquad \bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap \cdots \cap A_n$$

(Here the notation $:=$ means that the left hand side of the equality is being defined as a shorthand for the right hand side.)

The operators $\bigcup_{k=1}^{n}$ and $\bigcap_{k=1}^{n}$ act as variable-binding operators on the variable k. The expression $\bigcup_{k=1}^{n} A_k$ is meant to be read as: "the union of the sets A_k where k ranges from 1 to n" and $\bigcap_{k=1}^{n} A_k$ the same with intersection instead of union.

Proposition 5.2

Explicitly, for any x from the domain of discourse, we have:

$$x \in \bigcup_{k=1}^{n} A_k \quad \Leftrightarrow \quad \exists k \in \{1, \cdots, n\}, x \in A_k$$
$$x \in \bigcap_{k=1}^{n} A_k \quad \Leftrightarrow \quad \forall k \in \{1, \cdots, n\}, x \in A_k$$

Proof. We have $\exists k_0 \in \{1, \cdots, n\}, x \in A_{k_0}$ if and only if $\bigvee_{k=1}^{n} (x \in A_k)$.

Similarly, $\forall k \in \{1, \cdots, n\}, x \in A_k$ if and only if $\bigwedge_{k=1}^{n} (x \in A_k)$. $\square$

For example, if $A_1 := \{1,2,3,4\}$, $A_2 := \{0,2,4\}$ and $A_3 := \{2,3,5\}$, then we have:

$$\bigcup_{k=1}^{3} A_k = A_1 \cup A_2 \cup A_3 = \{0,1,2,3,4,5\}$$
$$\bigcap_{k=1}^{3} A_k = A_1 \cap A_2 \cap A_3 = \{2\}$$

Notice the following inclusions:

Proposition 5.3

Let $A_1, \ldots, A_n$ be sets. Then:

$$\forall i \in \{1, \cdots, n\}, \quad A_i \subseteq \left(\bigcup_{k=1}^{n} A_k\right) \qquad \forall i \in \{1, \cdots, n\}, \quad \left(\bigcap_{k=1}^{n} A_k\right) \subseteq A_i$$

Proof. These inclusions are direct consequences of the proposition 5.2. □

While we could study in-depth these more general operations on any number of sets, we are going to favour stating properties for the initial definitions of the union and intersection of two sets because those results can be easily generalised to any number of sets thanks to the properties of the logical operators $\wedge$ and $\vee$ as well as their set-theoretic analogous $\cap$ and $\cup$ (notably the property of *associativity*, that is, the property stating, broadly speaking, that the order of operations does not matter).

The following proposition states that the union of any subsets of two sets A and B is a subset of the union $A \cup B$:

Proposition 5.4

Let A and B be sets. Then:

$$\forall X \subseteq A, \quad \forall Y \subseteq B, \quad (X \cup Y) \subseteq (A \cup B)$$

Proof. Let $X \subseteq A$ and $Y \subseteq B$. Let $x \in (X \cup Y)$. Then we have $x \in X$ or $x \in Y$. If $x \in X$, then $x \in A$ for X is a subset of A. Similarly, if $x \in Y$, then $x \in B$. Then $x \in A$ or $x \in B$ which by definition means $x \in (A \cup B)$. Thus $(X \cup Y) \subseteq (A \cup B)$. □

Corollary 5.1

Let A be a set. Then:

$$\forall X, Y \subseteq A, \quad (X \cup Y) \subseteq A$$

Proof. This is a special case of the previous proposition when $A = B$ combined with the fact that $A \cup A = A$ established in proposition 5.1. □

The following proposition gives an explicit form for the union of two sets defined intentionally:

Proposition 5.5

Let A and B be two sets. Let $P[x]$ and $Q[x]$ be two predicates.
Let $X := \{x \in A \mid P(x)\}$ and $Y := \{x \in B \mid Q(x)\}$. Then:

$$X \cup Y = \{x \in (A \cup B) \mid (x \in A \wedge P(x)) \vee (x \in B \wedge Q(x))\}$$

Proof. We know that $(X \cup Y) \subseteq (A \cup B)$ by proposition 5.4, justifying this choice of the set $(A \cup B)$ from which the elements are drawn.

Let x be an element of the domain of discourse.

$$\begin{array}{rrcl}
\text{By definition:} & x \in (X \cup Y) & \Leftrightarrow & (x \in X \;\;\vee\;\; x \in Y) \\
\text{But, on one hand:} & x \in X & \Leftrightarrow & (x \in A \;\;\wedge\;\; P(x)) \\
\text{On the other hand:} & x \in Y & \Leftrightarrow & (x \in B \;\;\wedge\;\; Q(x)) \\
\text{By equivalence:} & x \in (X \cup Y) & \Leftrightarrow & ((x \in A \;\;\wedge\;\; P(x)) \;\;\vee\;\; (x \in B \;\;\wedge\;\; Q(x)))
\end{array}$$

□

Corollary 5.2

With the same notations, in the common case where $A = B$, we have:

$$X \cup Y = \{x \in A \mid P(x) \vee Q(x)\}$$

Similar to what we have shown for the union, we have the following proposition:

Proposition 5.6

Let A and B be two sets. Let $P[x]$ and $Q[x]$ be two predicates.
Let $X := \{x \in A \mid P(x)\}$ and $Y := \{x \in B \mid Q(x)\}$. Then:

$$X \cap Y = \{x \in (A \cap B) \mid P(x) \wedge Q(x)\}$$

Proof. Let x be an element of the domain of discourse.

$$\begin{array}{rrcl}
\text{By definition:} & x \in (X \cap Y) & \Leftrightarrow & (x \in X \;\;\wedge\;\; x \in Y) \\
\text{But, on one hand:} & x \in X & \Leftrightarrow & (x \in A \;\;\wedge\;\; P(x)) \\
\text{On the other hand:} & x \in Y & \Leftrightarrow & (x \in B \;\;\wedge\;\; Q(x)) \\
\text{By equivalence:} & x \in (X \cap Y) & \Leftrightarrow & (x \in A \;\;\wedge\;\; P(x) \;\;\wedge\;\; x \in B \;\;\wedge\;\; Q(x)) \\
\text{But, by definition:} & x \in A \;\;\wedge\;\; x \in B & \Leftrightarrow & x \in (A \cap B) \\
\text{Thus:} & x \in (X \cap Y) & \Leftrightarrow & (x \in (A \cap B) \;\;\wedge\;\; P(x) \;\;\wedge\;\; Q(x))
\end{array}$$

□

Corollary 5.3

With the same notations, in the common case where $A = B$, we have:

$$X \cap Y = \{x \in A \mid P(x) \wedge Q(x)\}$$

We can characterise the inclusion relation with unions and intersections:

Proposition 5.7

Let A and B be sets. Then we have the equivalences:

$$B \subseteq A \;\Leftrightarrow\; (A \cup B) \subseteq A \;\Leftrightarrow\; B \subseteq (A \cap B)$$

Proof. ▪ Suppose $B \subseteq A$. Let $x \in (A \cup B)$. Then by definition $x \in B$ or $x \in A$. But since $B \subseteq A$, then necessarily $x \in A$. Thus $A \cup B \subseteq A$.

▪ Conversely, assume $A \cup B \subseteq A$. From the proposition 5.3 we deduce $B \subseteq A \cup B$. Then from proposition 3.2 we conclude $B \subseteq A$.

▪ Assume $B \subseteq A$. Let $x \in B$. Then by hypothesis, $x \in A$. Thus x is in both A and B which means by definition that $x \in A \cap B$. So we do have $B \subseteq A \cap B$.

▪ Conversely, suppose $B \subseteq A \cap B$. From proposition 5.3 we deduce $A \cap B \subseteq A$. Finally, from proposition 3.2 we have $B \subseteq A$. □

Note that, since the inclusions $A \subseteq A \cup B$ and $A \cap B \subseteq B$ are verified for any sets A and B as stated in proposition 5.3, we do not need further proof to establish the equivalences:

$$B \subseteq A \;\Leftrightarrow\; A = A \cup B \;\Leftrightarrow\; B = A \cap B$$

An important case for the intersection of two sets is given as follows:

Definition 5.3 : Disjoint sets

Let A and B be sets. The sets A and B are said to be **disjoint** if $A \cap B = \varnothing$.
In other words, they are disjoint if they share no element in common.

So far we have talked about the two sets operations known as union and intersection. Through the properties we listed about them, we have come to the conclusion that the set-theoretic union is analogous to the logical disjunction while the set-theoretic intersection is analogous to the logical conjunction. We have also seen that the set-theoretic inclusion relation resembles the logical implication in certain ways and that the equality between sets is similar to the logical equivalence. Remains only the logical negation that we have not associated with a set-theoretic notion yet. With that goal in mind, let us now introduce a new operation one can apply on two sets:

Definition 5.4 : Relative complement

Let A and B be two sets. We define the **relative complement** of A in B (or **set difference** of B and A) as the set, denoted $B \setminus A$, such that:

$$B \setminus A := \{x \in B \mid x \notin A\}$$

In other words, the elements of $B \setminus A$ are exactly the elements of B that are not in A.

For example, if:

- $A := \{1, 2\}$ and $B := \{0, 1, 2, 3\}$ then $B \setminus A = \{0, 3\}$
- $A := \{1, 2\}$ and $B := \{0, 2, 4\}$ then $B \setminus A = \{0, 4\}$

Let us now show the connection between the set-theoretic complement and the logical negation by means of intentionally defined sets:

Proposition 5.8

Let A and B be two sets. Let $P[x]$ and $Q[x]$ be two predicates.
Let $X := \{x \in A \mid P(x)\}$ and $Y := \{x \in B \mid Q(x)\}$. Then:

$$Y \setminus X = \{x \in Y \mid x \in A \Rightarrow \neg P(x)\}$$

Proof. Let x be an element of the domain of discourse.

$$
\begin{array}{rrcl}
\text{By definition:} & x \in Y \setminus X & \Leftrightarrow & (x \in Y \quad \wedge \quad x \notin X) \\
\text{But:} & x \notin X & \Leftrightarrow & (x \in A \Rightarrow \neg P(x)) \\
\text{Then:} & x \in Y \setminus X & \Leftrightarrow & (x \in Y \quad \wedge \quad (x \in A \Rightarrow \neg P(x)))
\end{array}
$$

□

The connection with the logical negation is then given by the following corollary:

Corollary 5.4

With the same notations, in the common case where $A = B$, we have:

$$Y \setminus X = \{x \in Y \mid \neg P(x)\}$$

Now let us establish some basic properties of the relative complement:

Proposition 5.9

Let A be a set. Then: $A \setminus \varnothing = A$ as well as $\varnothing \setminus A = \varnothing$.

Proof. $A \setminus \varnothing = \{x \in A \mid x \notin \varnothing\} = A$ because $x \notin \varnothing$ is a tautology.
$\varnothing \setminus A = \{x \in \varnothing \mid x \notin A\} = \varnothing$ because there exists no element to draw in $\varnothing$. □

Here is yet another characterisation for the inclusion relation in terms of the relative complement:

Proposition 5.10

Let A and B be two sets. Then: $B \setminus A = \varnothing$ if and only if $B \subseteq A$. In particular, $A \setminus A = \varnothing$.

Proof. If $B \setminus A = \varnothing$ this means by definition that $\{x \in B \mid x \notin A\} = \varnothing$. In other words, there exists no element $x \in B$ that are not in A, formally: $\neg(\exists x \in B, x \notin A)$. Which is equivalent to $\forall x \in B, x \in A$, which by definition means $B \subseteq A$. □

Proposition 5.11

Let A, B and C be sets. Then: $C \setminus (B \setminus A) = (C \setminus B) \cup (C \cap A)$.
In particular: $C \setminus (C \setminus A) = C \cap A$.

Proof. Let x be an element of the domain of discourse.

$$
\begin{array}{rrcl}
\text{By definition:} & x \in C \setminus (B \setminus A) & \Leftrightarrow & (x \in C \quad \wedge \quad x \notin B \setminus A) \\
\text{But:} & x \notin B \setminus A & \Leftrightarrow & (x \notin B \quad \vee \quad x \in A) \\
\text{By equivalence:} & x \in C \setminus (B \setminus A) & \Leftrightarrow & (x \in C \quad \wedge \quad (x \notin B \quad \vee \quad x \in A)) \\
\text{By distributivity:} & & \Leftrightarrow & ((x \in C \quad \wedge \quad x \notin B) \quad \vee \quad (x \in C \quad \wedge \quad x \in A)) \\
\text{By definitions:} & & \Leftrightarrow & (x \in C \setminus B \quad \vee \quad x \in (C \cap A)) \\
\text{By definition:} & x \in C \setminus (B \setminus A) & \Leftrightarrow & x \in (C \setminus B) \cup (C \cap A)
\end{array}
$$

The special case follows directly from the special case in proposition 5.10. □

In part II, we have introduced the De Morgan laws for the logical disjunction and conjunction with respect to the logical negation. We now establish the analogous laws for the set-theoretic union and intersection with respect to the relative complement:

Proposition 5.12 : De Morgan laws

Let $A_1, \ldots, A_n$ and B be sets. Then:

$$B \setminus \left(\bigcup_{k=1}^{n} A_k\right) = \bigcap_{k=1}^{n} (B \setminus A_k) \qquad B \setminus \left(\bigcap_{k=1}^{n} A_k\right) = \bigcup_{k=1}^{n} (B \setminus A_k)$$

Proof. Let x be an element of the domain of discourse.

$$\begin{array}{rrcl}
\text{By definition:} & x \in B \setminus \left(\bigcup_{k=1}^{n} A_k\right) & \Leftrightarrow & \left(x \in B \ \wedge \ x \notin \left(\bigcup_{k=1}^{n} A_k\right)\right) \\
\text{But by proposition 5.2:} & x \notin \left(\bigcup_{k=1}^{n} A_k\right) & \Leftrightarrow & \forall k \in \{1, \cdots, n\}, x \notin A_k \\
\text{Then:} & x \in B \setminus \left(\bigcup_{k=1}^{n} A_k\right) & \Leftrightarrow & (x \in B \ \wedge \ (\forall k \in \{1, \cdots, n\}, x \notin A_k)) \\
\text{By distributivity:} & & \Leftrightarrow & \forall k \in \{1, \cdots, n\}, x \in B \ \wedge \ x \notin A_k \\
\text{By definition:} & & \Leftrightarrow & \forall k \in \{1, \cdots, n\}, x \in B \setminus A_k \\
\text{Then by proposition 5.2:} & x \in B \setminus \left(\bigcup_{k=1}^{n} A_k\right) & \Leftrightarrow & x \in \bigcap_{k=1}^{n} (B \setminus A_k)
\end{array}$$

The same proof applies for the second set equality by switching the roles of $\bigcup$ and $\bigcap$ and $\forall$ by $\exists$. $\square$

Conversely, we now give equations for the relative complement of any set B in a set expressed under the form of an union or an intersection:

Proposition 5.13

Let $A_1, \ldots, A_n$ and B be sets. Then:

$$\left(\bigcup_{k=1}^{n} A_k\right) \setminus B = \bigcup_{k=1}^{n} (A_k \setminus B) \qquad \left(\bigcap_{k=1}^{n} A_k\right) \setminus B = \bigcap_{k=1}^{n} (A_k \setminus B)$$

Proof. Let x be an element of the domain of discourse.

$$\begin{array}{rcl}
x \in \left(\bigcup_{k=1}^{n} (A_k \setminus B)\right) & \Leftrightarrow & \exists k \in \{1, \cdots, n\}, (x \in A_k \ \wedge \ x \notin B) \\
& \Leftrightarrow & (\exists k \in \{1, \cdots, n\}, x \in A_k) \ \wedge \ x \notin B \\
x \in \left(\bigcup_{k=1}^{n} (A_k \setminus B)\right) & \Leftrightarrow & x \in \left(\bigcup_{k=1}^{n} A_k\right) \setminus B
\end{array}$$

The proof is the same for the other equality by switching the roles of $\bigcup$ by $\bigcap$ and $\exists$ by $\forall$. $\square$

In general, the domain of discourse is not strictly speaking a set. However, it sometimes can be a set: then any set we have at our disposal are subsets of that set. It also allows us to define another kind of complement (that happen to be a special case of the relative complement):

Definition 5.5 : Absolute complement

Let U be the set representing the domain of discourse (here called **universe**). Let A be a set (thus it is a subset of U).
We define the **absolute complement** of A, or simply **complement** of A, denoted by A^c or sometimes $\overline{A}$, as the set of elements in U that are not in A. Formally:

$$A^c := U \setminus A$$

All the results given for the relative complement also applies for the absolute complement.
Some results specific to the absolute complement are given here:

Proposition 5.14

Let A be a set. The complement of the complement of A is A, *i.e.*: $(A^c)^c = A$.

Proof. We have $A^c = U \setminus A$. So $(A^c)^c = U \setminus (U \setminus A)$. But by proposition 5.11 we then have $(A^c)^c = U \cap A$. Since $A \subseteq U$, according to proposition 5.7, then $U \cap A = A$, from which the result follows directly. □

Proposition 5.15

The absolute complement of the empty set is the universe and *vice versa*: $\varnothing^c = U$ and $U^c = \varnothing$.

Proof. We have $\varnothing^c = U \setminus \varnothing$. From proposition 5.9, then $\varnothing^c = U$. By proposition 5.14, it follows that $U^c = \varnothing$. □

Proposition 5.16 : De Morgan laws

Let A and B be two sets. Then: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

Proof. By definition: $(A \cup B)^c = U \setminus (A \cup B)$. From proposition 5.12, $U \setminus (A \cup B) = (U \setminus A) \cap (U \setminus B)$ which can be rewritten as $(A \cup B)^c = A^c \cap B^c$.
Similarly, $(A \cap B)^c = U \setminus (A \cap B) = (U \setminus A) \cup (U \setminus B) = A^c \cup B^c$. □

Proposition 5.17

Let A be a set. Then: $A \cap A^c = \varnothing$ and $A \cup A^c = U$.

Proof. We have: $A \cap A^c = \{x \in U \mid x \in A \wedge x \notin A\}$. Since the admissibility condition is a contradiction, the set is empty. By the De Morgan law and proposition 5.14, one obtains $(A \cap A^c)^c = A^c \cup A = A \cup A^c$. But $A \cap A^c = \varnothing$ so, by proposition 5.15, $A \cup A^c = U$. □

Proposition 5.18

Let A and B be sets. Then: $A \subseteq B \;\Leftrightarrow\; B^c \subseteq A^c$.

Proof. Suppose $A \subseteq B$. Let $x \in B^c$. Then $x \notin B$. But $x \in B$ is a necessary condition for $x \in A$ since $A \subseteq B$. Thus by contrapositive, one obtains $x \notin A$ which can be rewritten as $x \in A^c$. Thus $A \subseteq B \Rightarrow B^c \subseteq A^c$. The converse is verified by using the result we just proved and the fact that $(B^c)^c = B$ and $(A^c)^c = A$. □

Exercise

Show that, if A, B and C are sets, then:

$$(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C) \qquad A \text{ and } B \text{ are disjoint} \;\Leftrightarrow\; A \setminus B = A$$

So far we have been working on sets, that is, unordered collections of unique elements (possibly infinitely many of them). We would like now to introduce a new kind of object: tuples. They are to be understood as finite ordered collections of elements (possibly containing the same element multiple times). To define tuples of arbitrary length, we will first introduce the notion of an ordered pair in order to generalise afterwards:

Definition 5.6 : Ordered pair

Let A and B be two sets. Let $a \in A$ and $b \in B$ be two elements.
The **ordered pair** (or **couple**) formed by the two elements a and b, that we denote by (a, b), is the mathematical object verifying the following property:

$$\forall c \in A, \quad \forall d \in B, \quad (a, b) = (c, d) \;\Leftrightarrow\; (a = c) \wedge (b = d)$$

The elements a and b are called respectively first and second **components** (or **coordinates**).

We could formalise this new notion with the aid of some set-theoretic concepts but it would not bring much to the table considering the scope of this document. We would rather like the reader to remember this preceding property stating that two ordered pairs are equal if and only if their respective first and second components are equal.

To denote the set of all ordered pairs for given sets we introduce the following set operation:

Definition 5.7 : Cartesian product

Let A and B be sets. We define the **Cartesian product** of A and B, denoted $A \times B$, as the set of all ordered pairs whose first component is in A and second component is in B. Formally:

$$\forall x, \quad x \in (A \times B) \overset{\text{def}}{\Leftrightarrow} \exists a \in A, \exists b \in B, \quad x = (a, b)$$

For example, if $A := \{1, 2\}$ and $B := \{3, 4, 5\}$, then:

$$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$$

One important distinction of the Cartesian product of two sets with the union and intersection is the lack of *associativity*: if given 3 sets A, B and C, the sets $(A \times B) \times C$ and $A \times (B \times C)$ are *a priori* not equal. Take for example: $A := \{1, 2\}$, $B := \{3, 4\}$ and $C := \{5, 6\}$. Then an element of $(A \times B) \times C$ would be $((1, 3), 5)$: an ordered pair whose first component is an ordered pair and second component a number. While an element of $A \times (B \times C)$ would be $(1, (3, 5))$: also an ordered pair but its first component is a number and its second component is an ordered pair. We thus show, through that example, that the order in which the Cartesian products are performed impact the resulting set. This is in clear contrast with the intersection and union of two sets. Therefore, if we wish to generalise the Cartesian product to an arbitrary number of sets we will not be able to just rely on the Cartesian product on two sets as we did for the intersection and the union; we will need to define it on its own.

To do so, let us generalise this idea of ordered collection of elements to any number n of elements.

Definition 5.8 : Tuples

Let $A_1, A_2, \cdots, A_n$ be sets. Let $a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n$.
The n-**tuple** (or simply tuple) formed by the elements $a_1, a_2, \cdots, a_n$, that we denote by $(a_1, a_2, \cdots, a_n)$, is the mathematical object verifying the following property:

$$\forall b_1 \in A_1, \quad \forall b_2 \in A_2, \quad \cdots, \quad \forall b_n \in A_n$$

$$(a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n) \;\Leftrightarrow\; \forall i \in \{1, \cdots, n\}, a_i = b_i$$

The element a_k is called the k'th **component** (or **coordinate)** of the tuple.

Note that we could formalise these new objects using ordered pairs or even set-theoretic notions but, once again, this would not be of great interest considering the goal of this document. Note also that, for $n = 2$, we find the definition of the ordered pair again. For $n = 2$ we also say **couple**, $n = 3$ **triple**, $n = 4$ **quadruple**, etc.

Now let us define the Cartesian product for an arbitrary number of sets:

Definition 5.9 : Cartesian product

Let $A_1, A_2, \cdots, A_n$ be sets.
The **Cartesian product** of $A_1, A_2, \cdots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set consisting of all tuples whose k'th component is in A_k. Formally:

$$x \in (A_1 \times A_2 \times \cdots \times A_n) \overset{\text{def}}{\Leftrightarrow} \exists a_1 \in A_1, \exists a_2 \in A_2, \cdots, \exists a_n \in A_n, x = (a_1, a_2, \cdots, a_n)$$

We also denote it: $\displaystyle\prod_{k=1}^{n} A_k := A_1 \times A_2 \times \cdots \times A_n$.

Notice that this notation $\prod_{k=1}^{n} A_k$, meant to be read as "the Cartesian product of the sets A_k where k ranges from 1 to n", acts as a variable-binding operator on the variable k.

Here is an example of Cartesian product of three sets. If $A := \{1, 2\}$, $B := \{3, 4\}$ and $C := \{5, 6\}$:

$$A \times B \times C = \{(1,3,5), (1,3,6), (1,4,5), (1,4,6), (2,3,5), (2,3,6), (2,4,5), (2,4,6)\}$$

A special notation is used when performing the Cartesian product on the same set. Let A be a set and n a natural number then:

$$A^n := \prod_{k=1}^{n} A$$

Taking the same example $A := \{1, 2\}$, we have:

$$A^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$$

Proposition 5.19

Let A, B and C be sets. Then: $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

Proof. Let x be an element of the domain of discourse. We have the following equivalences:

$$\begin{aligned} x \in (A \cup B) \times C &\Leftrightarrow \exists a \in A \cup B, \exists c \in C, x = (a, c) \\ &\Leftrightarrow \exists a, (a \in A \vee a \in B) \wedge (\exists c \in C, x = (a, c)) \\ &\Leftrightarrow \exists a, (a \in A \wedge \exists c \in C, x = (a, c)) \vee (a \in B \wedge \exists c \in C, x = (a, c)) \\ x \in (A \cup B) \times C &\Leftrightarrow x \in (A \times C) \cup (B \times C) \end{aligned}$$

□

Exercise

Show that, if A, B and C are sets, then:

$$(A \cap B) \times C = (A \times C) \cap (B \times C) \qquad (A \setminus B) \times C = (A \times C) \setminus (B \times C)$$

To recap, so far we have talked about how to define sets and how to construct sets from preexisting sets by means of set operations such as the union, the intersection, the complements and the Cartesian products.

6 Well-known sets of numbers

Arguably, the most well-known sets are the sets consisting of numbers. We want to introduce them here without constructing them formally, as it would go beyond the scope of this document.

We can distinguish between different types of numbers: starting with the **natural numbers** also known as **whole numbers** or **non-negative integers**. These are the most simple numbers: they are used to count things. The set of all natural numbers is denoted by $\mathbb{N}$. We have:

$$\mathbb{N} := \{0, 1, 2, \cdots\}$$

It is often that we want to exclude the case for a variable to be 0 ranging over the natural numbers, then we introduce the following notation:

$$\mathbb{N}^* := \mathbb{N} \setminus \{0\}$$

If we want to consider the "negative integers" as well, we use the set of **integers**, denoted $\mathbb{Z}$:

$$\mathbb{Z} := \{\cdots, -2, -1, 0, 1, 2, \cdots\}$$

Same as for the natural numbers, if we want to exclude 0, then we write:

$$\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$$

Note that the set of natural numbers $\mathbb{N}$ is a subset of the set of integers $\mathbb{Z}$. In fact, $\mathbb{Z}$ is exactly the union of $\mathbb{N}$ and the set of "additive inverses" of the natural numbers. The additive inverse of a natural number $n \in \mathbb{N}$ is an integer $m \in \mathbb{Z}$ such that $n + m = 0$. We usually denote m by writing $-n$.

In $\mathbb{Z}$ we can add, subtract and multiply numbers, however we cannot divide as we wish. To do so, we introduce the set $\mathbb{Q}$ of **rational numbers** which consists of the fractions of integers. An element $x \in \mathbb{Q}$ is represented by two integers $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^*$ such that $x = \frac{n}{m}$. Formally:

$$x \in \mathbb{Q} \overset{\text{def}}{\Leftrightarrow} \exists n \in \mathbb{Z}, \exists m \in \mathbb{Z}^*, x = \frac{n}{m}$$

The integer n is called **numerator** while the integer m is called **denominator**. Note that $\mathbb{Z}$ is a subset of $\mathbb{Q}$ as any integer n can be written as the fraction $\frac{n}{1}$. We could have chosen $\mathbb{N}^*$ for the denominator as $\frac{n}{-m} = \frac{-n}{m}$.

We could in some sense understand the set $\mathbb{Q}$ as the Cartesian product $\mathbb{Z} \times \mathbb{N}^*$. However, it must be remembered that two fractions $\frac{n_1}{m_1}$ and $\frac{n_2}{m_2}$ can be equal even when $n_1 \neq n_2$ and $m_1 \neq m_2$. Thus, the Cartesian product $\mathbb{Z} \times \mathbb{N}^*$ is "too big" compared to $\mathbb{Q}$. We would need to add the constraint that n and m are coprime for instance. But we do not concern ourselves with the actual construction here.

Once again we introduce the notation:

$$\mathbb{Q}^* := \mathbb{Q} \setminus \{0\}$$

Numbers can be written in a *base*, for instance in base 10: this means we represent a number by using a succession of digits from 0 to 9. For example, some natural numbers and integers written in base 10 are: 3, 42, -5, -78. To write rational numbers in base 10 it is needed to use a **decimal point** that allows the quantity represented to not be a whole number. For instance: $\frac{1}{2} = 0.5$, $-\frac{7}{5} = -1.4$, $\frac{1}{3} = 0.33333\cdots$. That last example shows that the digits after the decimal point can possibly go on indefinitely. However, it can be proven that the rational numbers are exactly the numbers that either have a finite base 10 representation or, if they have a base 10 representation that goes on indefinitely, then at some point after the decimal point the digits repeat the same pattern indefinitely (for example, $\frac{9}{44} = 0.20454545\cdots$).

This leaves unchecked the numbers that go on indefinitely after the decimal point but whose digits do not end up repeating the same pattern. We call these numbers: **irrational numbers**. Examples of irrational numbers include $\sqrt{2}$, π and e.

We define the set of **real numbers**, denoted by $\mathbb{R}$, as the union of the set of rational numbers $\mathbb{Q}$ with the set of irrational numbers. In other words, $\mathbb{R}$ is the set of all numbers that can be written as a number in base 10 such that its decimal expansion is finite or infinite, repeating or not.

We introduce the following notations:

$$\mathbb{R}^* := \mathbb{R} \setminus \{0\} \qquad \mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geqslant 0\} \qquad \mathbb{R}_- := \{x \in \mathbb{R} \mid x \leqslant 0\}$$
$$\mathbb{R}_+^* := \{x \in \mathbb{R} \mid x > 0\} \qquad \mathbb{R}_-^* := \{x \in \mathbb{R} \mid x < 0\}$$

The real numbers can be represented geometrically as the **real number line**: it is a line on which we fix the number 0 in some place, then for any position on that line, the distance between that spot and 0 is a real number, and conversely, every real number can be associated with exactly one position on the line.

Notice that we have the following inclusion relations:

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

7 Recap

In this document, we have introduced a great deal of notions. Here are the ones we would like the reader to remember:

- **Sets** are collections of **elements**. They are determined *extensionally*, that is two sets are equal if and only if they consist of **exactly the same elements**. We have defined the notations $\forall x \in A$ and $\exists x \in A$ as well as Roster notation to denote simple sets by enumerating their elements. The **empty set** is the set with no elements.

- We have talked about the **inclusion relation** and **subsets**, that is sets contained within other sets. We have introduced the **power set** of a set. We then have established a very important theorem characterising the equality of sets in terms of a **double inclusion**.

- We then discussed another way to define subsets: with an **intentional** definition, which consists in specifying the *logical condition* $P[x]$ elements, of a given superset A, need to satisfy in order to belong to the subset. To denote such subsets we use the set-builder notation: $\{x \in A \mid P(x)\}$. We then showed the connection between the inclusion relation and the logical implication as well as the connection between the set equality relation and the logical equivalence.

- After that, we discussed another way to construct sets: using **set operations**. We introduced the **union** and **intersection** of two sets. After establishing basic properties of these operations, we have come across the *associativity* property they satisfy which in turn enabled us to define the union and the intersection of any arbitrary finite number of sets. We then showed the links between the set-theoretic union, the existential quantifier $\exists$ and the logical disjunction as well as between the set-theoretic intersection, the universal quantifier $\forall$ and the logical conjunction. After that we introduced the relative and absolute **complements** as new set operations and highlighted the connection between them and the logical negation. Then, we addressed a last set operation called the **Cartesian product** of two sets which consists of the **ordered pairs** whose components are in those sets. Lacking the associativity property, we thus had to define explicitly, using the new notion of **tuples**, the Cartesian product for an arbitrary finite number of sets.

- Finally, we introduced the well-known sets of numbers:
  - the set of the **natural numbers** or **whole numbers**, denoted by the symbol $\mathbb{N}$
  - the set of the **integers**, denoted by $\mathbb{Z}$
  - the set of the **rational numbers**, denoted by $\mathbb{Q}$
  - the set of the **real numbers**, denoted by $\mathbb{R}$

References

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